

Delegation Uncertainty*

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Abstract

Delegation bears an intrinsic form of uncertainty. Investors hire managers for their superior models of asset markets, but delegation outcome is uncertain precisely because managers' model is unknown to investors. We model investors' delegation decision as a trade-off between asset return uncertainty and delegation uncertainty. Our theory explains several puzzles on fund performances. It also delivers asset pricing implications supported by our empirical analysis: (1) because investors partially delegate and hedge against delegation uncertainty, CAPM alpha arises; (2) the cross-section dispersion of alpha increases in uncertainty; (3) managers bet on alpha, engaging in factor timing, but factors' alpha is immune to the rise of their arbitrage capital – when investors delegate more, delegation hedging becomes stronger. Finally, we offer a novel approach to extract model uncertainty from asset returns, delegation, and survey expectations.

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1 Introduction

The asset management industry is being revolutionized by exploding data sources and increasingly sophisticated techniques for data analysis. Big data help asset managers construct and estimate models of asset returns. However, data collection and analysis require expertise, creating a division of knowledge between asset managers and investors. As a result, delegation outcome is uncertain even in the absence of moral hazard, because it depends on asset managers' superior models of asset returns that are unknown to investors.

We model two types of agents: managers who know the return distribution, and investors who face model uncertainty (ambiguity) given by a set of probability distributions (“models”). Investors pay managers to allocate part of their wealth, and allocate the retained wealth under ambiguity. We abstract away from moral hazard.¹ Here managers dutifully use their probability knowledge to allocate the delegated wealth on the efficient frontier.

Delegation improves investors' welfare by reducing their exposure to ambiguity in the returns of individual assets. As in [Gennaioli, Shleifer, and Vishny \(2015\)](#), such welfare view resolves important puzzles in the asset management literature. For example, we characterize conditions under which delegation happens even when managers underperform the market or deliver zero alpha by holding portfolios proportional to the market.

However, delegation uncertainty remains – even if managers deliver the efficient portfolio, the efficient frontier varies across probability models. Investors incorporate such uncertainty in their delegation decision and hedge delegation uncertainty when allocating the retained wealth. Their portfolio tilts towards assets whose returns move against the frontier across models. A benchmark for our analysis is CAPM – the equilibrium without model uncertainty. Our equilibrium deviates from CAPM due to investors' delegation hedging. The cross-section dispersion of CAPM alpha increases in model uncertainty. Importantly, alpha is immune to the rise of arbitrage capital, i.e., the wealth allocated by managers, because when investors delegate more, their hedging against delegation uncertainty is stronger.

In our model, professional asset managers and investors are different in their knowledge of return distribution. To highlight such division of knowledge, we assume that managers cannot inform investors of the true return distribution, and that investors do not learn the probability distribution by observing managers' allocation in asset markets.²

¹[Miao and Rivera \(2016\)](#) study the principal-agent problem in a similar setup of heterogeneous belief.

²Learning under model uncertainty (ambiguity) has been studied by [Epstein and Schneider \(2007\)](#), and

We provide closed-form solutions for investors’ delegation and the cross section of expected asset returns by solving a quadratic approximation of investors’ utility under ambiguity.³ As a technical contribution, our approximation extends that of [Maccheroni, Marinacci, and Ruffino \(2013\)](#) into functional spaces. When delegation is unavailable, and investors are ambiguity-neutral, our approximation becomes the classic Arrow-Pratt approximation, which generates the mean-variance portfolio of [Markowitz \(1959\)](#) and a CAPM equilibrium.

In our setup, delegation offers investors *model-contingent* allocation of wealth. Asset managers can be viewed as portfolio formation machines with the knowledge of true return distribution as input and the corresponding efficient portfolio as output. In investors’ mind, the overall structure of uncertainty is a two-step lottery: first, a probability model is drawn and observed by managers who allocate the delegated wealth on the efficient frontier; second, a state of the world is drawn according to the probability model. Therefore, the delegation portfolio is model-contingent, and the delegation return is both state- and model-contingent.

Delegation improves investors’ welfare by offering access to efficient allocation under each probability model – whichever model is true, managers know it and allocate efficiently. However, delegation does not eliminate ambiguity. It transforms ambiguity from the returns of individual assets to that of the efficient frontier. Investors’ optimal delegation depends on the trade-off between the welfare gains from such transformation and the management fees.⁴

This new perspective on delegation explains the puzzling findings that investors delegate in spite of unconvincing performances of managers. Investors cannot evaluate performances under rational expectation, so econometricians’ performance measurements are based on an information set different from investors’. How delegation improves welfare depends on the structure of model uncertainty. We characterize conditions under which delegation arises even if managers underperform the market, deliver negative alpha after fees, or simply hold portfolios proportional to the market portfolio ([Fama and French \(2010\)](#); [Lewellen \(2011\)](#)).

Investors are averse to the cross-model comovement between asset returns and the

in the asset pricing literature, [Leippold, Trojani, and Vanini \(2008\)](#), [Ju and Miao \(2012\)](#) and [Choi \(2016\)](#). [Mele and Sangiorgi \(2015\)](#) study agents’ information acquisition under Knightian uncertainty. [Pástor and Stambaugh \(2012\)](#) study how investors’ Bayesian learning affects their delegation decision.

³We assume smooth ambiguity aversion utility function proposed by [Klibanoff, Marinacci, and Mukerji \(2005\)](#) and [Nau \(2006\)](#) and discussed by [Epstein \(2010\)](#) and [Klibanoff, Marinacci, and Mukerji \(2012\)](#). [Ghirardato, Maccheroni, and Marinacci \(2004\)](#) take an axiomatic approach to study the separation between ambiguity and agents’ attitude towards ambiguity.

⁴Appendix IV illustrates how our framework can be used as a normative model to guide the delegation choice of investors under model uncertainty. Management fees represent managers’ effort costs, agency cost, search and screening costs, relative bargaining power, or other types of inefficiencies not explicitly modeled.

delegation (i.e., frontier) return. They hedge against delegation uncertainty, which induces a two-factor structure in the expected asset returns: a standard CAPM market risk premium, and a model uncertainty premium (“alpha”) that increases in the level of delegation and model uncertainty. Specifically, investors favor assets whose returns move against that of the frontier across probability models. Such assets have low or even negative alpha. Investors avoid assets whose returns comove with that of the frontier. Such assets have high alpha.

One would expect assets’ alpha to converge to zero if the economy approaches full delegation (e.g., driven by declining management fees), because managers with mean-variance portfolios almost dominate the asset markets. However, this is not the case. The more investors delegate, the stronger they hedge against delegation uncertainty per dollar of retained wealth. The increasing hedging motive counter-balances the decreasing share of wealth managed by investors themselves, which sustains the CAPM alpha. Therefore, our model sheds light on why certain investment strategies still deliver alpha in spite of the growing arbitrage capital, i.e., the money managed by professionals who know those “anomalies”.

Our model delivers other asset pricing implications. The market risk premium declines in the level of delegation, which suggests that as the asset management sector grows, the security market line will be increasingly flat. Following [Bewley \(2011\)](#), we simplify investors’ model uncertainty by relating it to the statistical errors in parameter estimation. The overall level of ambiguity and investors’ sentiment, which is directly mapped to survey expectations, emerge as the key determinants of the cross-section variation of asset returns.⁵ This simplified setup is later used to extract ambiguity from asset returns, delegation, and survey data.

We test the model assumptions and asset pricing implications using the U.S. equity factors that are well studied in the literature of empirical asset pricing. We use factors rather than individual stocks because a parsimonious factor structure largely spans stock returns.⁶

The main prediction of the model is that assets deliver low CAPM alphas if they are viewed by investors as insurance against delegation uncertainty. Measuring investors’ subjective belief is challenging, so we take a revealed-preference approach. Following an increase of model uncertainty, investors overweigh delegation-hedging assets. Therefore, we

⁵Our model does not feature limits to arbitrage, but due to delegation uncertainty, investors’ sentiment survives in the expected asset returns even if the level of wealth professionally managed approaches 100%.

⁶Among a large set of firm characteristics that have been proposed to predict returns in the cross section, [Hou, Xue, and Zhang \(2015\)](#) show that a four-factor model summarizes the cross section of average returns, [Freyberger, Neuhierl, and Weber \(2017\)](#) identify a small subset that provide distinct information, and [Kozak, Nagel, and Santosh \(2017\)](#) find the principal components approximate the stochastic discount factor well.

calculate the correlation between a factor’s individual-investor ownership and measures of uncertainty. High-correlation assets are revealed by investors’ choices as the delegation-hedging assets, and consistent with the model’s prediction, they have low CAPM alpha.

Next, we characterize how the cross section of factors’ CAPM alphas vary over time. Our model predicts that the cross-section dispersion increases in periods of higher uncertainty because investors engage in stronger delegation hedging. We find a strong correlation between the alpha dispersion and various uncertainty measures.

The key assumption of our model is that managers have superior knowledge of return distribution. Since the cross section of factors vary over time, we should observe managers tilting portfolios towards factors with desirable distributional properties. We sort factors by their institutional ownership (*INST*), and find those with high *INST* outperform those with low *INST*. Parametric tests confirm this finding: one standard-deviation increase of *INST* adds 1.76% (annualized) to a factor’s future return, which is a 53% increase over the average factor return in our sample. High-*INST* factors also have high Sharpe ratios.

Finally, we extract investors’ model uncertainty from data by fitting the equilibrium conditions of asset markets directly to the factor returns, delegation, and survey of investors’ expectations (Greenwood and Shleifer (2014)). The model-implied uncertainty exhibits cyclical dynamics and peaks around market turmoils, such as the dot-com bubble and the Great Recession. Our measure exhibits comovement with alternative uncertainty measures in the literature but contains distinct information. The correlation ranges from 0.11 to 0.51.

Literature. Our paper furthers the studies on ambiguity, i.e., the lack of knowledge of probability distribution (Knight (1921)). Ellsberg paradox is an example of ambiguity aversion.⁷ Widely cited as a challenge to the expected utility theory (e.g., Dow and Werlang (1992)), ambiguity aversion has been introduced in various fields in economics, such as asset pricing (e.g., Boyarchenko (2012), Cao, Wang, and Zhang (2005), Chen and Epstein (2002), Epstein and Wang (1994), Garlappi, Uppal, and Wang (2007), Horvath (2016), Maenhout (2004), Illeditsch (2011), Ilut (2012), Ju and Miao (2012)), real option (Miao and Wang (2011)), corporate governance (Izhakian and Yermack (2017)), market microstructure (Condie and Ganguli (2011); Easley and O’Hara (2010); Ozsoylev and Werner (2011); Vitale (2018)), and policy intervention in crises (Caballero and Krishnamurthy (2008)). Epstein (2010),

⁷A version of it was noted by John Maynard Keynes in his book "A Treatise on Probability" (1921).

Guidolin and Rinaldi (2010), and Hansen and Sargent (2016) review the literature.

Our setup is a special case of the multi-agent environments discussed by Hansen and Sargent (2012). Here one type of agents (investors) face ambiguity while the other (managers) do not. Closely related, Miao and Rivera (2016) study the corporate finance implications of optimal contracting between a principal, who faces ambiguity, and an agent, who knows the probability. We differ by abstracting away moral hazard and focusing instead on the asset pricing implications of delegation uncertainty that is intrinsic to the division of knowledge. The incentive problems of delegation under ambiguity are also studied by Fabretti, Herzel, and Pinar (2014) and Rantakari (2008). Hirshleifer, Huang, and Teoh (2017) study whether investors' market participation can be improved by introducing funds whose allocation is contingent upon ambiguous asset supply. We differ by emphasizing investors' hedging against delegation uncertainty, and its implications on the cross-section variation of asset returns.

Our results are purely driven by the belief heterogeneity between investors and managers. Managers are endowed with informational advantage, and unlike Bhattacharya and Pfleiderer (1985), investors know managers' ability. In the recent literature, Gârleanu and Pedersen (2018), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), Huang (2016), Sockin and Zhang (2018) study fund managers' information acquisition. Pástor and Stambaugh (2012) model investors' learning of the value-added from delegation. We also abstract away incentive problems of fund managers, which has been studied in a large literature (e.g., Basak and Pavlova (2013), Binsbergen, Brandt, and Koijen (2008), Buffa, Vayanos, and Woolley (2014), Chevalier and Ellison (1999), Cuoco and Kaniel (2011), Dow and Gorton (1997), Guerrieri and Kondor (2012), He and Xiong (2013), Heinkel and Stoughton (1994), Kaniel and Kondor (2013), Leung (2014), Starks (1987), Ou-Yang (2003)). Moreover, our results do not rely on fund flow dynamics (Berk and Green (2004)) or the heterogeneity in fund characteristics (Pástor, Stambaugh, and Taylor (2017)).

While the economic mechanism is simple, our model helps understand a variety of puzzling findings. Since Jensen (1968), a large literature has documented that asset managers fail to beat passive benchmarks or deliver "alpha" (e.g., Barras, Scaillet, and Wermers (2010), Carhart (1997), Del Guercio and Reuter (2014), Fama and French (2010), Gruber (1996), Malkiel (1995), Wermers (2000)). Specifically, Fama and French (2010) find that the aggregate portfolio of actively managed U.S. equity funds is close to the market portfolio (see also Lewellen (2011)), and few funds produce sufficient benchmark-adjusted returns to cover

their costs. Nevertheless, the asset management sector has been growing dramatically. Following [Gennaioli, Shleifer, and Vishny \(2015\)](#), we propose an alternative perspective based on subjective welfare, and characterize the conditions under which managers underperform the market, deliver negative alpha after fees, and hold portfolios proportional to market.

This paper contributes to the asset pricing literature by characterizing a hedging demand that arises from the heterogeneity of probability knowledge. Ambiguity hedging is also emphasized by [Drechsler \(2013\)](#), who show that investors pay a premium for index options to hedge model misspecification. Moreover, we decompose assets' expected returns into risk and ambiguity premia, with the prices of risk and ambiguity dependent on the endogenous level of delegation. The ambiguity premium is positive even if investors are not ambiguity-averse, which is in stark contrast to the existing literature (e.g., [Brenner and Izhakian \(2017\)](#), [Epstein and Schneider \(2008\)](#), [Kogan and Wang \(2003\)](#), [Trojani and Vanini \(2004\)](#), [Ui \(2011\)](#)). Through delegation, investors' return on wealth is both state- and model-contingent, so even ambiguity-neutral investors cannot average out model uncertainty for each state of the world, acting as expected-utility agents. Investors have to deal with the joint uncertainty in both the state and model space. We are the first to show that delegation arises from ambiguity, and at the same time, fundamentally changes its role in agents' decision making.⁸

On the empirical side, we find that the cross section of factors' returns and CAPM alphas is partly spanned delegation uncertainty, and that its dispersion varies with the level of uncertainty. Moreover, we provide new evidence on factor timing ([Cohen, Polk, and Vuolteenaho \(2003\)](#); [Moreira and Muir \(2017\)](#)). Related to our revealed-preference approach, [Greenwood and Hanson \(2012\)](#) use firms' equity issuance decisions to time factors, assuming that firm managers have superior information. Finally, our paper offers new evidence on the relationship between institutional ownership and factor premia. [Nagel \(2005\)](#) find the unconditional factor premia are most pronounced among stocks with low institutional ownership. We find that the conditional factor premia increase in institutional ownership.

Finally, we use the model's asset-market equilibrium conditions to back out ambiguity from investor survey. Our approach is related to [Bhandari, Borovička, and Ho \(2016\)](#) who use macroeconomic models to extract ambiguity shocks from survey data on households' expectations about inflation and unemployment. Based on the theory of [Izhakian \(2014\)](#),

⁸We show that delegation transforms the ambiguity on individual assets to that on the efficient frontier, and its implications on asset pricing. [Uppal and Wang \(2003\)](#) emphasize different types of ambiguity (the overall ambiguity and that on a subset of assets) and study the implications on under-diversification.

Brenner and Izhakian (2017) extract investors’ ambiguity from intraday data of stock prices.

2 Model

2.1 The Setup

The economy has N risky assets, a risk-free asset with return r_f , and a unit mass of representative investors. Each investor is matched with a fund manager. Agents make decisions at date 0. Asset returns are realized at date 1. The vector of asset returns, $\mathbf{r} = \{r_i\}_{i=1}^N$, is a mapping from Ω , the set of states of the world at date 1, to real numbers, $\mathbf{r} : \Omega \mapsto \mathbf{R}^N$.

Endowed with one unit of wealth, an investor chooses δ , the fraction of wealth invested in the fund, and allocates the retained wealth $1 - \delta$ according to \mathbf{w}^o , a column vector of portfolio weights on the N risky assets (superscript “o” for the investor’s “own” portfolio). The penniless fund manager chooses \mathbf{w}^d , the delegation portfolio.

Information and preference. The investor makes decisions under ambiguity (model uncertainty). A non-singleton set, Δ , contains candidate probability distributions of \mathbf{r} (“models”). For $Q \in \Delta$, the investor assigns a prior $\pi(Q)$ of Q being the true model.

The investor’s preference is represented by the smooth ambiguity-averse utility function in Klibanoff, Marinacci, and Mukerji (2005). It separates ambiguity from the attitude toward ambiguity, which is important for our analysis.⁹ The utility is defined over the investor’s terminal wealth, $r_{\delta, \mathbf{w}^o, \mathbf{w}^d}$, whose subscripts show the dependence on delegation δ , the investor’s own portfolio \mathbf{w}^o , and the delegation portfolio \mathbf{w}^d . The utility is

$$V(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) = \int_{\Delta} \phi \left(\int_{\Omega} u(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) dQ(\omega) \right) d\pi(Q) \quad (1)$$

$\phi(\cdot)$ and $u(\cdot)$ are strictly increasing functions and twice continuously differentiable. The concavities of $u(\cdot)$ and $\phi(\cdot)$ capture risk and ambiguity aversion respectively.

The fund manager knows the true model, denoted by P , and acts as a portfolio formation machine that delivers the corresponding efficient portfolio, $w^d(P)$. We will specify $w^d(P)$ after introducing the quadratic approximation of investor utility. To access this “ma-

⁹Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012) discuss the experimental evidence.

chine”, the investor pays a proportional management fee ψ .¹⁰ While existing models typically assume that managers have better information on the first moment by obtaining return signals, here managers’ skill is in a general form of distribution knowledge. Busse (1999) finds volatility-timing by mutual funds (see Chen and Liang (2007) for hedge funds).¹¹ Jondeau and Rockinger (2012) calculate the welfare improvements from general distribution timing. We provide our own evidence on managers’ distribution knowledge in Section 3.

Delegation as model-contingent allocation. From the investor’s perspective, for any $Q \in \Delta$, if it is the true model, the manager knows it and constructs the *corresponding* efficient portfolio, $\mathbf{w}^d(Q)$. Therefore, delegation makes the investor’s wealth *model-contingent*:

$$\begin{aligned} r_{\delta, \mathbf{w}^o, \mathbf{w}^d} &= (1 - \delta) \left[r_f + (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right] + \delta \left[r_f + (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(Q) \right] \\ &= r_f + (\mathbf{r} - r_f \mathbf{1})^T \left[(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d(Q) \right], \quad Q \in \Delta. \end{aligned} \quad (2)$$

The investor’s own portfolio is a N -dimensional *vector*, $\mathbf{w}^o \in \mathbf{R}^N$. In contrast, the delegation portfolio, \mathbf{w}^d , is a *mapping* from the model space to real numbers, $\mathbf{r} : \Delta \mapsto \mathbf{R}^N$. Through delegation, the return on wealth becomes a mapping from the state *and* model spaces to real numbers, $r_{\delta, \mathbf{w}^o, \mathbf{w}^d} : \Omega \times \Delta \mapsto \mathbf{R}$. Without delegation (i.e., $\delta = 0$), the return is given by $r_f + (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o$, which is just a mapping from the state space, Ω , to \mathbf{R} .

As in Segal (1990), let us consider an imaginary economy with two stages: (1) the investor chooses δ and \mathbf{w}^o ; (2) the probability model is drawn and known by the manager who allocates the delegated wealth. Therefore, delegation allows investors to achieve efficient allocations under each possible model. However, delegation uncertainty remains – the efficient portfolio, $\mathbf{w}^d(Q)$, varies across probability models. The manager cannot inform the investor which model is true; otherwise, the delegation uncertainty disappears. This captures the realistic obstacles in the communication between managers and investors.

2.2 Quadratic Approximation

To solve the investor’s delegation and portfolio allocation in closed forms, we approximate the utility function in a quadratic fashion by extending the results of Maccheroni, Marinacci,

¹⁰In richer settings, ψ can be determined by the competition among managers, the competition among investors, managers’ costs of effort and information acquisition, investors’ search costs, agency cost, etc.

¹¹Ferson and Mo (2016) propose to measure to evaluate managers by both market and volatility timing.

and Ruffino (2013, MMR) into functional spaces. MMR does not allow agents' wealth to be model-contingent. We adopt their technical regularity conditions.

Definition 1 *A representative investor's certainty equivalent is defined by*

$$C(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) = v^{-1} \left(\int_{\Delta} \phi \left(\int_{\Omega} u(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) dQ(\omega) \right) d\pi(Q) \right), \quad (3)$$

where v is a composite function $v = \phi \circ u$.

The investor's delegation and portfolio problem is given by

$$\max_{\mathbf{w}^o, \delta} \{C(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) - \psi \delta\} \quad (4)$$

where $r_{\delta, \mathbf{w}^o, \mathbf{w}^d}$ is the return on wealth (Equation (2)) and ψ is the asset management fee.

We define two parameters, risk aversion and ambiguity aversion, respectively in a small neighborhood of the return on wealth around the risk-free rate r_f .

Definition 2 *At risk free return r_f , the local absolute risk aversion γ is defined as*

$$\gamma = -\frac{u''(r_f)}{u'(r_f)} \quad (5)$$

and marginal-utility-adjusted, local ambiguity aversion θ is defined as

$$\theta = -u'(r_f) \frac{\phi''(u(r_f))}{\phi'(u(r_f))} \quad (6)$$

To present the quadratic approximation of investor utility, we introduce several notations. We denote the excess return of a portfolio, \mathbf{w} , by $R^{\mathbf{w}} = (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}$, and its expectation under $Q \in \Delta$, by $R_Q^{\mathbf{w}} = E_Q \left[(\mathbf{r} - r_f \mathbf{1})^T \mathbf{w} \right]$. For a random variable X and given $Q \in \Delta$, let $E_Q(X)$ and $\sigma_Q^2(X)$ denote the expectation and variance respectively if X is a scalar, and μ_Q^X and Σ_Q^X denote the expectation vector and the covariance matrix respectively if X is a vector. Given $Q \in \Delta$, the covariance of two random variables, X and Y , is $cov_Q(X, Y)$. We define the investor's average model under the prior, π ,

$$\bar{Q}(A) = \int_{Q \in \Delta} Q(A) d\pi(Q), \text{ for any } A \subset \Omega. \quad (7)$$

Following MMR, we approximate the certainty equivalent using the Taylor expansion in the space of portfolio weights. Since the delegation portfolio weights, $\mathbf{w}^d(Q)$, are functionals defined on the model (probability) space, our Taylor expansion relies on the generalized Fréchet derivatives in the Banach spaces. The proof is provided in Appendix I.

Proposition 1 (Quadratic Preference) *The smooth ambiguity-averse preference over the state- and model-contingent return, $r_{\delta, \mathbf{w}^o, \mathbf{w}^d}$, is represented by the certainty equivalent,*

$$\begin{aligned} C(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) = & r_f + (1 - \delta) R_{\bar{Q}}^{\mathbf{w}^o} - \frac{(1 - \delta)^2}{2} \left[\gamma \sigma_{\bar{Q}}^2(R^{\mathbf{w}^o}) + \theta \sigma_{\pi}^2(R_{\bar{Q}}^{\mathbf{w}^o}) \right] + \\ & \delta E_{\pi} \left(R_Q^{\mathbf{w}^d} \right) - \frac{\delta^2}{2} \left[\gamma E_{\pi} \left(\sigma_Q^2 \left(R^{\mathbf{w}^d} \right) \right) + \theta \sigma_{\pi}^2 \left(R_Q^{\mathbf{w}^d} \right) \right] \\ & - (\theta + \gamma) (1 - \delta) \delta \text{cov}_{\pi} \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) + \mathbf{R}(\mathbf{w}^o, \mathbf{w}^d), \end{aligned} \quad (8)$$

where $\mathbf{R}(\mathbf{w}^o, \mathbf{w}^d)$ is a high-order term that satisfies $\lim_{(\mathbf{w}^o, \mathbf{w}^d) \rightarrow \mathbf{0}} \frac{\mathbf{R}(\mathbf{w}^o, \mathbf{w}^d)}{\|(\mathbf{w}^o, \mathbf{w}^d)\|^2} = 0$.

As in MMR, the residual term can be ignored if portfolios are sufficiently diversified with matrix/ L^2 norms close to zero. The approximation allows us to intuitively understand the investor's preference. The utility increases in $R_{\bar{Q}}^{\mathbf{w}^o}$, the \bar{Q} -expected excess return on the investor's retained wealth, and its sensitivity, $(1 - \delta)^2$, decreases in the level of delegation δ . The utility decreases in $\sigma_{\bar{Q}}^2(R^{\mathbf{w}^o})$, the \bar{Q} -variance of excess return on retained wealth, and the risk sensitivity increases in γ , the risk aversion. The utility decreases in $\sigma_{\pi}^2(R_{\bar{Q}}^{\mathbf{w}^o})$, which measures ambiguity, i.e., the *cross-model* variation of the expected excess return, $R_{\bar{Q}}^{\mathbf{w}^o}$. The ambiguity sensitivity increases in θ , the ambiguity aversion. As investors delegate more, i.e., δ increases, the sensitivities to risk and ambiguity of return on retained wealth both decline.

The delegation return enters into the utility in a similar fashion. The utility increases in $E_{\pi} \left(R_Q^{\mathbf{w}^d} \right)$, which is the expected excess return from delegation averaged over models,

$$E_{\pi} \left(R_Q^{\mathbf{w}^d} \right) = \int_{Q \in \Delta} E_Q \left[(\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(Q) \right] d\pi(Q),$$

where $R_Q^{\mathbf{w}^d}$ is the Q -expected excess return from delegation. The utility decreases in $\sigma_{\pi}^2 \left(R_Q^{\mathbf{w}^d} \right)$, which measures ambiguity, i.e., the cross-model variation of expected excess return from delegation, and its sensitivity increases in θ , the ambiguity aversion. The utility decreases in $E_{\pi} \left(\sigma_Q^2 \left(R^{\mathbf{w}^d} \right) \right)$, a measure of risk averaged by π over models, and its sensitivity increases

in γ , the risk aversion. The sensitivities to delegation ambiguity and risk both increase in δ .

The terms discussed so far can be summarized into two categories: (1) the expected returns and return variances and covariances (“risk”) averaged over models; (2) the cross-model variance of the expected returns (“ambiguity”). The approximation shows how these statistics enter into utility through risk aversion, ambiguity aversion, and delegation.

The last term deserves more attention. The cross-model covariance between the expected delegation return and the expected return on retained wealth enters the investor’s utility with a negative sign. It captures a cross-model hedging motive. When allocating the retained wealth, the investor prefers assets that deliver high expected returns under models where the expected delegation return is low. The utility’s sensitivity is maximized at $\delta = \frac{1}{2}$. Intuitively, the comovement between the delegation performance and the investor’s own investment matters the most when her wealth is split 50/50.

Finally, we show that our quadratic approximation nests MMR’s solution and the mean-variance utility as special cases.

Corollary 1 *Without delegation, i.e., $\delta = 0$, the approximation degenerates to the quadratic approximation in [Maccheroni, Marinacci, and Ruffino \(2013\)](#):*

$$C \left(r_f + (\mathbf{r} - r_f \mathbf{1})^T [(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d(Q)] \right) \approx r_f + R_{\bar{Q}}^{\mathbf{w}^o} - \frac{\gamma}{2} \sigma_{\bar{Q}}^2 (R^{\mathbf{w}^o}) - \frac{\theta}{2} \sigma_{\pi}^2 \left(R_{\bar{Q}}^{\mathbf{w}^o} \right). \quad (9)$$

Without delegation and ambiguity aversion, i.e., $\delta = 0$ and $\theta = 0$, our approximation degenerates to the mean-variance utility under the average model \bar{Q} :

$$r_f + R_{\bar{Q}}^{\mathbf{w}^o} - \frac{\gamma}{2} \sigma_{\bar{Q}}^2 (R^{\mathbf{w}^o}). \quad (10)$$

Delegation portfolio. The investor informs her risk aversion, γ , to the manager who forms a mean-variance efficient portfolio given his knowledge of the true model. In the investor’s mind, for any $Q \in \Delta$, if it is the true model, the managers solves

$$\max_{\mathbf{w}^d} \left\{ (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T \mathbf{w}^d - \frac{\gamma}{2} (\mathbf{w}^d)^T \Sigma_Q^{\mathbf{r}} (\mathbf{w}^d) \right\}.$$

The delegation portfolio is a mapping from the model space to real numbers, $\mathbf{w}^d : \Delta \mapsto \mathbf{R}^N$,

$$\mathbf{w}^d(Q) = (\gamma \Sigma_Q^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1}). \quad (11)$$

2.3 Investor Optimization

We solve the optimal level of delegation δ and the investor's portfolio \mathbf{w}^o by maximizing the quadratic approximation given by Equation (8). Details are provided in Appendix II.

Proposition 2 (Optimal Delegation) *Given the optimal portfolio \mathbf{w}^o , the investor's optimal delegation level δ is given by*

$$\delta = \frac{E_\pi \left(R_Q^{\mathbf{w}^d} \right) - R_Q^{\mathbf{w}^o} - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) - \psi}{E_\pi \left(R_Q^{\mathbf{w}^d} \right) - R_Q^{\mathbf{w}^o} - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right)}. \quad (12)$$

Delegation increases if the delegation return is expected to be high across models (high $E_\pi \left(R_Q^{\mathbf{w}^d} \right)$), and if it does not vary a lot across models (low $\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right)$). Delegation decreases if the investor achieves a high return on her own, (high $R_Q^{\mathbf{w}^o}$), and if across models, the expected return on retained wealth comoves closely with the expected delegation return (high $\text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right)$). The investor are averse to such cross-model comovement.

Proposition 3 (Investor Portfolio) *Given the optimal level of delegation δ , the investor's own portfolio of risky assets is given by*

$$\mathbf{w}_\delta^o = \left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_{\bar{Q}}^{\mathbf{r}}} \right)^{-1} \left[\left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) - \underbrace{(\theta + \gamma) \left(\frac{\delta}{1 - \delta} \right) \text{cov}_\pi \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right)}_{\text{Delegation Hedging}} \right]. \quad (13)$$

Without delegation, the investor's portfolio is $\left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_{\bar{Q}}^{\mathbf{r}}} \right)^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right)$, which is MMR's solution.¹² Risk is measured by $\Sigma_{\bar{Q}}^{\mathbf{r}}$, the covariance matrix of asset returns under the average model \bar{Q} . It is scaled by γ , the risk aversion. Ambiguity is measured by $\Sigma_\pi^{\mu_{\bar{Q}}^{\mathbf{r}}}$, the cross-model covariance of *expected* asset returns, $\mu_{\bar{Q}}^{\mathbf{r}}$. It is scaled by θ , the ambiguity aversion. If $\theta = 0$, the portfolio degenerates to the formula of [Markowitz \(1959\)](#) under \bar{Q} .

When $\delta > 0$, the portfolio exhibits a hedging demand from $\text{cov}_\pi \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right)$, the cross-model covariance between the expected asset returns, $\mu_{\bar{Q}}^{\mathbf{r}}$, and the expected delegation return, $R_Q^{\mathbf{w}^d}$. The investor knows that whichever model is true, the manager knows it and constructs the efficient portfolio accordingly, but the true model is still unknown. Therefore, the investor designs her own portfolio to hedge such ambiguity.

¹²[Garlappi, Uppal, and Wang \(2007\)](#) derive a similar portfolio by incorporating estimation errors in expected returns as a source of ambiguity (a maxmin approach as in [Gilboa and Schmeidler \(1989\)](#)).

The hedging demand does not disappear even if we shut down ambiguity aversion ($\theta = 0$). The intuition can be explained by inspecting an ambiguity-neutral investor's utility,

$$V(r_{\omega,Q}) = \int_{Q \in \Delta} \int_{\omega \in \Omega} u(r_{\omega,Q}) dQ(\omega) d\pi(Q),$$

where the subscripts of return on wealth, ω and Q , highlight that the return is state-dependent, and through delegation, model-contingent. An ambiguity-neutral investor cannot perform Bayesian model averaging and operates under \bar{Q} , but instead, has to deal with the *joint* uncertainty of state and model. Therefore, the cross-model covariance between the expected asset returns and the expected delegation return still appears in investors' portfolio choice even if the investor is risk-averse (i.e., having concave $u(\cdot)$) but not ambiguity-averse.

Let $cov_{\pi}(\mu_Q^{r_i}, R_Q^{w^d})$ denote the i -th element of $cov_{\pi}(\mu_Q^r, R_Q^{w^d})$. It is the cross-model covariance between asset i 's expected return and the delegation return. When $cov_{\pi}(\mu_Q^{r_i}, R_Q^{w^d}) > 0$, the investor reduces exposure to asset i ; When $cov_{\pi}(\mu_Q^{r_i}, R_Q^{w^d}) < 0$, the investor tilts her portfolio towards asset i , buying an insurance against delegation uncertainty. Next we explore the implications of delegation hedging on the cross-sectional variation of asset returns.

2.4 The Cross-Section of Asset Returns

We characterize the cross section of expected asset returns and their CAPM alpha. First, we show that when delegation is unavailable, our model reproduces the key theoretical findings in the current asset pricing literature. Next, we show how delegation changes the results.

To understand the impact of ambiguity, and in particular, delegation uncertainty, on the cross section of asset returns, we use CAPM as the natural benchmark. Here, when asset returns follow normal distributions and $u(\cdot)$ is the CARA (constant absolute risk aversion) utility, the delegation portfolio, $\mathbf{w}^d(Q)$, maximizes the expected utility. Therefore, without ambiguity, investors and managers both choose the mean-variance portfolio and the asset-market equilibrium is CAPM. Ambiguity causes the deviations from CAPM.

Equilibrium without delegation. We define the market portfolio \mathbf{m} , which is the sum of investors' and managers' asset demands and is also equal to the exogenous asset supply:

$$\mathbf{m} = \delta \mathbf{w}^d(P) + (1 - \delta) \mathbf{w}^o. \quad (14)$$

When delegation is unavailable, the investor's portfolio is given by

$$\mathbf{w}_0^o = \left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right)^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right), \quad (15)$$

where the subscript “0” is for “zero-delegation”. Since $\delta = 0$, substituting $\mathbf{m} = \mathbf{w}_0^o$ into Equation (15) and multiplying both sides by $\left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right)$, we have

$$\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} = \left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right) \mathbf{m}. \quad (16)$$

Note that $\Sigma_{\bar{Q}}^{\mathbf{r}} \mathbf{m}$ is simply the vector of covariance under \bar{Q} between the asset returns and the market return, and $\Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \mathbf{m}$ is the covariance under π between the *expected* asset returns and the *expected* market return. The former measures risk while the latter measures ambiguity. If investors' average model is true, i.e., $\bar{Q} = P$, the left-hand side is the assets' expected excess returns under the true probability measure, and the right-hand side offers a decomposition.

Proposition 4 (Ambiguity Premium without Delegation) *When delegation is unavailable, the equilibrium expected excess returns of risky assets are*

$$\mu_P^{\mathbf{r}} - r_f \mathbf{1} = \lambda_{\mathbf{m}} \boldsymbol{\beta}_{\mathbf{r}, \mathbf{m}}^P + \lambda_{\mathbf{w}_0^o} \boldsymbol{\beta}_{\mu_{\bar{Q}}^{\mathbf{r}}, \mathbf{m}}^{\pi}, \quad (17)$$

if investors' average model is the true model, i.e., $\bar{Q} = P$. Here we define: (1) the market price of risk, $\lambda_{\mathbf{m}} = \gamma \sigma_P^2(R^{\mathbf{m}})$, and the risk beta, $\boldsymbol{\beta}_{\mathbf{r}, \mathbf{m}}^P = \frac{\text{cov}_P(\mathbf{r}, R^{\mathbf{m}})}{\sigma_P^2(R^{\mathbf{m}})}$; (2) The market price of ambiguity, $\lambda_{\mathbf{w}_0^o} = \theta \sigma_{\pi}^2(R_Q^{\mathbf{m}})$, and the ambiguity beta, $\boldsymbol{\beta}_{\mu_{\bar{Q}}^{\mathbf{r}}, \mathbf{m}}^{\pi} = \frac{\text{cov}_{\pi}(\mu_{\bar{Q}}^{\mathbf{r}}, R_Q^{\mathbf{m}})}{\sigma_{\pi}^2(R_Q^{\mathbf{m}})}$.

Equation (17) decomposes the expected excess returns. The first component is the standard CAPM beta multiplied by the standard price of risk (the return variance scaled by γ). The second component is an *ambiguity premium*. The ambiguity beta measures the cross-model comovement between the expected asset returns and the return of investors' (market) portfolio. If asset i has a positive beta (i.e., $\beta_{\mu_{\bar{Q}}^{\mathbf{r}}, \mathbf{m}}^{\pi} > 0$), it delivers a higher average return. If asset i has a negative beta, it serves as a hedge against model uncertainty and delivers a lower average return. Ambiguity beta is priced at $\lambda_{\mathbf{w}_0^o} = \theta \sigma_{\pi}^2(R_Q^{\mathbf{m}})$, which is the total amount of ambiguity in the expected market return scaled by θ , the ambiguity aversion.

The assumption of $\bar{Q} = P$ is important. Under ambiguity, investors cannot evaluate the assets' expected returns under the true model. Instead, they evaluate assets by averaging

over models, and require fair compensations for risk and ambiguity under \bar{Q} . Only if $\bar{Q} = P$, the expected returns under investors' average model, $\mu_{\bar{Q}}^r$, coincide with the expected returns under the true model, which are observed by econometricians through the average returns. Otherwise we cannot solve μ_P^r using the optimality condition on investor's portfolio choice.

The ambiguity premium is CAPM alpha as in MMR. They analyze a special case of two assets where one has pure risk (known distribution) while the other bears ambiguity. Kogan and Wang (2003) derive the similar decomposition of expected returns using the constrained-robust approach. In those models and here, if we shut down ambiguity aversion, the price of ambiguity, $\lambda_{\mathbf{w}_0} = \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}_0} \right)$, is zero, and the model degenerates to CAPM.

Corollary 2 (CAPM without Delegation) *When delegation is unavailable and investors are ambiguity-neutral ($\theta = 0$), the expected excess returns of risky assets are given by*

$$\mu_P^r - r_f \mathbf{1} = \lambda_{\mathbf{m}} \beta_{\mathbf{r}, \mathbf{m}}^P. \quad (18)$$

if the investors' average model is the true model (i.e., $\bar{Q} = P$).

If the investor is ambiguity-neutral, the investor's utility function can be written as

$$V(r_\omega) = \int_{\Delta} \int_{\omega \in \Omega} u(r_\omega) dQ(\omega) d\pi(Q) = \int_{\omega \in \Omega} u(r_\omega) \left[\int_{\Delta} dQ(\omega) d\pi(Q) \right] = \int_{\omega \in \Omega} u(r_\omega) d\bar{Q}(\omega),$$

where the subscript ω of return on wealth highlights the fact that the return is only state-dependent. The investor behaves as an expected-utility agent under \bar{Q} and chooses the standard mean-variance portfolio under the quadratic utility, so if $\bar{Q} = P$, we rediscover CAPM. It is critical that $u(r)$ can be taken out of the integral operator, \int_{Δ} , on the model space, because r is not model-dependent. Once delegation is available, r is both state- and model-dependent, so the equilibrium deviates from CAPM even without ambiguity aversion.

Equilibrium with delegation. When delegation is available, the market portfolio is equal to a mixture of managers' portfolio and investors' portfolio, i.e., $\mathbf{m} = \delta \mathbf{w}^d(P) + (1 - \delta) \mathbf{w}^o$. We arrange the fund manager's portfolio, $\mathbf{w}^d(P) = (\gamma \Sigma_P^r)^{-1} (\mu_P^r - r_f \mathbf{1})$, under the true probability distribution P , and arrive at the following expression of expected excess returns:

$$\mu_P^r - r_f \mathbf{1} = (\gamma \Sigma_P^r) \mathbf{w}^d(P).^{13} \quad (19)$$

¹³Note that because μ_P^r already shows up in managers' portfolio, we do not need to assume $\bar{Q} = P$ to

Using the rearranged market clearing condition, $\mathbf{w}^d(P) = \frac{1}{\delta}\mathbf{m} - \left(\frac{1-\delta}{\delta}\right)\mathbf{w}^o$, we rewrite (19),

$$\mu_P^r - r_f\mathbf{1} = \underbrace{\frac{1}{\delta}\gamma\Sigma_P^r\mathbf{m}}_{\text{CAPM Component}} + \underbrace{\left(\frac{1-\delta}{\delta}\right)\gamma\Sigma_P^r(-\mathbf{w}^o)}_{\text{CAPM Alpha}}. \quad (20)$$

Using the definition of market beta in Proposition (4), we can write the first term on the right-hand side as the product of assets' market beta, $\beta_{\mathbf{r},\mathbf{m}}^P = \frac{\text{cov}_P(\mathbf{r}, R^m)}{\sigma_P^2(R^m)}$, and a new price of risk, $\lambda_\delta = \frac{\gamma}{\delta}\sigma_P^2(R^m)$, which now depends on delegation. Specifically, an increase in δ leads a decrease in the market price of risk, i.e., a flatter security market line. This property is in line with the concurrence of a growing asset management industry and a declining equity premium in the U.S. market (documented by Jagannathan, McGrattan, and Scherbina (2001), Lettau, Ludvigson, and Wachter (2008) among others).

Substituting investors' portfolio (Equation (3)) into the second component, we have

$$\boldsymbol{\alpha} = \gamma\Sigma_P^r \left(\gamma\Sigma_Q^r + \theta\Sigma_\pi^{\mu_Q^r} \right)^{-1} \left[\underbrace{(\theta + \gamma)\text{cov}_\pi\left(\mu_Q^r, R_Q^{\mathbf{w}^d}\right)}_{\text{Delegation Hedging}} - \underbrace{\left(\frac{1-\delta}{\delta}\right)\left(\mu_Q^r - r_f\mathbf{1}\right)}_{\text{Average Belief}} \right], \quad (21)$$

which has one component from investors' hedging against delegation uncertainty and the other from investors' average belief of expected returns ("sentiment"). A high sentiment is associated with a low ambiguity premium. This component disappears if δ approaches 100%.

The other component from delegation hedging is immune to the changes of delegation level. As δ approaches 100%, the hedging motive becomes increasingly strong as shown by the coefficient, $\left(\frac{\delta}{1-\delta}\right)$, of $\text{cov}_\pi\left(\mu_Q^r, R_Q^{\mathbf{w}^d}\right)$ in investors' portfolio (Equation (13)). Therefore, even if investors manage less wealth when δ increases, they hedge more *per unit* of retained wealth. When we substitute investors' portfolio into Equation (20), this coefficient exactly offsets $\left(\frac{1-\delta}{\delta}\right)$, the ratio of retained-to-delegated wealth, in the CAPM alpha.

Proposition 5 (Delegation and Ambiguity Premium) *The expected excess returns of risky assets are given by*

$$\mu_P^r - r_f\mathbf{1} = \lambda_\delta\beta_{\mathbf{r},\mathbf{m}}^P + \boldsymbol{\alpha}. \quad (22)$$

The market betas, $\beta_{\mathbf{r},\mathbf{m}}^P$, are defined in Proposition 4. The price of risk, λ_δ , is $\frac{\gamma}{\delta}\sigma_P^2(R^m)$. The CAPM alpha is given by Equation (21), which depends investors' average belief, $\left(\mu_Q^r - r_f\mathbf{1}\right)$,

solve the equilibrium expected returns as we did for the case without delegation.

and the cross-model covariance the assets' expected returns and the expected delegation return, $cov_\pi(\mu_Q^r, R_Q^{w^d})$. When δ approaches 100% (due to declining management fees or changes in model uncertainty), α converges to $\gamma \Sigma_P^r (\gamma \Sigma_Q^r + \theta \Sigma_\pi^{\mu_Q^r})^{-1} (\theta + \gamma) cov_\pi(\mu_Q^r, R_Q^{w^d})$.

When δ is precisely equal to 100%, we have $\mathbf{m} = \mathbf{w}^d(P)$, and CAPM reemerges:

$$\mu_P^r - r_f \mathbf{1} = \beta_{\mathbf{r}, \mathbf{m}}^P \lambda_{\mathbf{m}}, \quad (23)$$

where $\lambda_{\mathbf{m}} = R_P^{w^d} = R_P^{\mathbf{m}}$. However, as long as $\delta < 100\%$, investors need to allocate their retained wealth under ambiguity. The more they delegate, the stronger they hedge against delegation uncertainty. Therefore, even if the wealth managed by investors declines and the wealth allocated on the mean-variance frontier rises, the increasingly strong hedging demand of investors sustains the CAPM alpha and generates a discontinuity at the limit of $\delta = 100\%$.

Interestingly, even if we may shut down ambiguity aversion, i.e., $\theta = 0$, the delegation-hedging component of ambiguity premium still exists. This property distinguishes our model from the existing models that feature zero ambiguity premium if agents are not ambiguity-averse. As in the discussion of investors' portfolio choice (Equation (13)), here the intuition can be explained by inspecting an ambiguity-neutral investor's utility function,

$$V(r_{\omega, Q}) = \int_{Q \in \Delta} \int_{\omega \in \Omega} u(r_{\omega, Q}) dQ(\omega) d\pi(Q).$$

Due to delegation, the return on wealth is both state- and model-dependent. As a result, the investor cannot perform Bayesian model averaging as she does in the case without delegation, and has to deal with the *joint* uncertainty of state and model. Therefore, $cov_\pi(\mu_Q^r, R_Q^{w^d})$, the hedging motive, appears in investors' portfolio and the ambiguity premium even if $\theta = 0$.

In the past few decades, asset management industry has grown dramatically, especially in the area of quantitative investment that targets alphas identified in the academic literature. Many have argued that the strategies' alphas shrink as arbitrage capital increases (e.g., [McLean and Pontiff \(2016\)](#)). Yet many strategies survive, and we will show an example in Section 3. Together they constitute a rich set of "anomalies" in asset pricing. Our model sheds light on such phenomena. Professional asset managers obtain arbitrage capital mainly through investors' delegation. As delegation increases, investors' hedging against delegation uncertainty becomes stronger, which sustains the CAPM alpha.

2.5 Delegation and Fund Performances

Simplifying model uncertainty. Here we simplify the structure of investors’ model uncertainty to derive intuitive comparative statics for the optimal delegation and characterize conditions under which delegation happens in spite of funds underperforming the market. We will also solve asset pricing conditions under the simplified model uncertainty that can be directly mapped to data for our empirical analysis in Section 3.

We make three assumptions that lead to typical settings of delegation – managers and investors differ in the knowledge of first moments of return distribution. For example, managers may receive return signals that improve the precision of expected-return estimates.

Assumption 1 *The investor knows the true covariance matrix: for any $Q \in \Delta$, $\Sigma_Q^r = \Sigma_P^r$.*

Given the quadratic approximation of investor utility, the relevant model uncertainty is now only in the expected returns, which is captured by the cross-model covariance, $\Sigma_\pi^{\mu_Q^r}$ under the prior π .¹⁴ This case of known covariance and unknown expected returns echoes the observation by Merton (1980). Kogan and Wang (2003) also consider this case in their study of portfolio selection under ambiguity. The next assumption links ambiguity to volatility.

Assumption 2 *The investor’s subjective belief over the expected returns is given by a normal distribution, whose covariance is proportional to the true return variance:*

$$\mu_Q^r \sim N\left(\mu_Q^r, v\Sigma_P^r\right). \quad (24)$$

Since $\mu_Q^r \sim N\left(\mu_Q^r, v\Sigma_P^r\right)$, v that parameterizes the level of model uncertainty. This setup echoes the interpretation of ambiguity as statistical errors by Bewley (2011).¹⁵ We may interpret v as the inverse of sample size. If the investor has T observations of \mathbf{r}_t , the method-of-moment estimator of the expected return is $\frac{1}{T}\sum_{t=1}^T \mathbf{r}_t$ and its covariance is $\frac{1}{T}\Sigma_P^r$. Therefore, $\Sigma_\pi^{\mu_Q^r} = v\Sigma_P^r$ with $v = \frac{1}{T}$. A larger v means a smaller sample and larger estimation errors. It is natural to assume that $v < 1$ because $\frac{1}{T} < 1$ for non-singleton samples.

Assumption 3 $v < 1$.

¹⁴Boyle, Garlappi, Uppal, and Wang (2012), and Ilut and Schneider (2014) introduce ambiguity through the uncertainty in the mean in models of financial markets and macroeconomy respectively.

¹⁵Bewley (2011) formulated the argument that confidence intervals are measures of the level of ambiguity associated with the estimated parameters. Easley and OHara (2010) adopt a similar formulation.

The normality assumption of the prior over $\mu_Q^{\mathbf{r}}$ brings technical convenience. Specifically, the expected delegation return can be decomposed as follows,

$$\begin{aligned} R_Q^{\mathbf{w}^d} &= (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T \mathbf{w}^d(Q) = (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1}) \\ &= \underbrace{(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})}_{\text{Chi-squared}} + 2 \underbrace{(\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})}_{\text{constant vector}} + \underbrace{R_Q^{\mathbf{w}^d}}_{\text{Normal}} + \underbrace{R_Q^{\mathbf{w}^d}}_{\text{constant}}, \end{aligned}$$

where the distributional properties are labeled below each term. Using Isserlis' theorem and the properties of Chi-squared and normal distributions, we solve in Appendix III the summary statistics in investors' optimal portfolio and delegation. In particular, we have

$$\text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right) = \frac{2v}{\gamma} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right), \quad (25)$$

so the strength of delegation hedging depends on the level of model uncertainty, v . This property helps directly map several model implications to data, for example, helping us back out investors' ambiguity from data on assets' CAPM residuals and delegation in Section 3.

Proposition 6 (Comparative Statics) *Under the three assumptions, the investor's portfolio is given by*

$$\mathbf{w}^o = \left(\frac{1}{\gamma + \theta v} \right) (\Sigma_P^{\mathbf{r}})^{-1} \left[\left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) - (\gamma + \theta) \left(\frac{\delta}{1 - \delta} \right) \frac{2v}{\gamma} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) \right]. \quad (26)$$

The optimal delegation decision is given by

$$\delta = \frac{\frac{v}{\gamma} N - \psi + \left[1 - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right) \right] R_Q^{\mathbf{w}^d}}{\left(1 + 2\frac{\theta v}{\gamma} \right) \frac{v}{\gamma} N + \left[1 + 4\frac{\theta v}{\gamma} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right)^2 \right] R_Q^{\mathbf{w}^d}}, \quad (27)$$

where $R_Q^{\mathbf{w}^d}$ is the expected delegation return under the average model, \bar{Q} . We have the following results of comparative statics:

- 1 The optimal level of delegation δ increases in N , the number of risky asset, i.e., $\frac{\partial \delta}{\partial N} > 0$.
- 2 The optimal level of delegation δ decreases in θ , the ambiguity aversion, v , the level of ambiguity, and ψ , the management fee, while increases in γ , the risk aversion, i.e., $\frac{\partial \delta}{\partial v} < 0$, $\frac{\partial \delta}{\partial \theta} < 0$, $\frac{\partial \delta}{\partial \psi} < 0$, and $\frac{\partial \delta}{\partial \gamma} > 0$, if N is sufficiently large.

3 Given the delegation level δ , investors' positions in risky assets, \mathbf{w}^o , decrease in θ , the ambiguity aversion, and v , the level of ambiguity, i.e., $\frac{\partial \mathbf{w}^o}{\partial v} < 0$ and $\frac{\partial \mathbf{w}^o}{\partial \theta} < 0$.

Under the simplified ambiguity, N , the number of assets, shows up in the investor's delegation and portfolio choices because, as previously discussed, the expected delegation return follows a Chi-squared distribution under the prior π and N appears in its mean and variance as the degree of freedom (details in Appendix III). Intuitively, as the number of risky assets increases, delegation brings more welfare improvements by transforming the ambiguity of many individual assets into the cross-model variation of a single efficient frontier.

We may interpret N as the number of risk factors instead of primitive assets. If there are an infinite number of assets with returns spanned by N factors and their idiosyncratic shocks, by the law of large numbers, investors can diversify away and ignore idiosyncratic shocks for any model as long as it is not a point-mass distribution. Back-of-envelope calculation shows that Equation (27) produces reasonable levels of delegation. If $N = 10$, $\gamma = 5$, $\theta = 1$, $R_{\frac{Q}{Q}}^{\mathbf{w}^d} = 4\%$, $\psi = 1\%$ and $v = 1/100$, we have $\delta = 49\%$. It rises to 99% if $N = 1000$.¹⁶

Holding N constant, delegation decreases in ambiguity aversion (θ) and the level of ambiguity (v), and increases in risk aversion (γ). The upside of delegation is that within a probability model, wealth is allocated efficiently, while the downside of delegation (aside from the management fee) is the exposure to delegation uncertainty, i.e., the cross-model variation of frontier. When model uncertainty and the aversion to it are higher, the latter dominates; when risk aversion is higher (and thus, it is more costly to be away from the frontiers), the former is more valued. The sum of portfolio weights on risky assets, i.e., $\mathbf{1}^T \mathbf{w}^o$, is the total risky investment. The investor becomes more conservative, when facing more ambiguity or having a higher level of ambiguity aversion.¹⁷

Fund performances. As reviewed by French (2008), the evidence on average fund performance suggests that investors are better off not delegating and instead holding indices. This poses a challenge to understand the growth of professional asset management in recent decades. In this paper, we shift the focus from ex post performance to ex ante welfare.

Performance measurement assumes that investors have the econometricians' belief (i.e., rational expectation). In reality and our model, investors face model uncertainty. Delegation

¹⁶In Appendix IV, we obtain the model uncertainty from a Bayesian learning problem, and illustrate how to incorporate it into the general delegation formula in Proposition 12.

¹⁷This is consistent with the comparative statics in settings without delegation (e.g., Gollier (2011)).

improves welfare by transforming ambiguity from individual assets to the frontier and making investors' delegated model-contingent. When choosing the level of delegation, the trade-off is between *within-model* allocation efficiency and *cross-model* delegation uncertainty.

Next, we compare the performance of funds and the market return, and show that even if the latter dominates under rational expectation, delegation may still be positive for investors under ambiguity. Substituting the investor's portfolio (equation (26)) into the market clearing condition, we solve the expected market excess return under the true model:

$$\begin{aligned} R_P^m &= \delta R_P^{\mathbf{w}^d} + (1 - \delta) R_P^{\mathbf{w}^o} \\ &= (\mu_P^{\mathbf{r}} - r_f \mathbf{1})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left[(\mu_P^{\mathbf{r}} - r_f \mathbf{1}) \delta + \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) \left(\frac{(1 - \delta) \gamma - \delta 2v (\theta + \gamma)}{\gamma + v\theta} \right) \right]. \end{aligned}$$

The expected excess return of the delegation portfolio is

$$R_P^{\mathbf{w}^d} = (\mu_P^{\mathbf{r}} - r_f \mathbf{1})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} (\mu_P^{\mathbf{r}} - r_f \mathbf{1})$$

The difference between the two, $R_P^{\mathbf{w}^d} - R_P^m$, is equal to

$$(1 - \delta) (\mu_P^{\mathbf{r}} - r_f \mathbf{1})^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left[(\mu_P^{\mathbf{r}} - r_f \mathbf{1}) - \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) \left(\frac{\gamma - \left(\frac{\delta}{1 - \delta} \right) 2v (\theta + \gamma)}{\gamma + v\theta} \right) \right], \quad (28)$$

which is also the average performance difference in a *large* sample.

Proposition 7 (Delegation and Underperformance) *Under the simplified ambiguity, fund managers underperform the market if*

$$\sum_{i=1}^N \mathbf{w}_i^d(P) (\mu_P^{\mathbf{r}_i} - r_f) < \varkappa \sum_{i=1}^N \mathbf{w}_i^d(P) \left(\mu_Q^{\mathbf{r}_i} - r_f \mathbf{1} \right),$$

where $\mathbf{w}_i^d(P)$ is the managers' portfolio weight on asset i , and the constant \varkappa is given by

$$\varkappa = \frac{\gamma - \left(\frac{\delta}{1 - \delta} \right) 2v (\theta + \gamma)}{\gamma + v\theta}, \quad (29)$$

which increases in θ and v and decreases in γ .

Whether the managers underperform or outperform the market depends on the comparison between the weighted-average of assets' expected returns under the true model and

that under the investors’ average model (scaled by \varkappa). Because investors also trade assets, managers’ performance depends on their relative aggression in risk- and ambiguity-taking. For example, if investors have in mind a high-return market (i.e. high μ_Q^r), they trade aggressively and earn a higher average return through more exposure to risk and ambiguity.

Therefore, in our model, delegation can arise in spite of managers’ underperformance relative to the market. Investors do not know the true model, so they cannot evaluate fund performances under rational expectation and choose between funds and the market index. Note that we do not impose any restriction on investors’ portfolio choice, so holding the market portfolio is certainly within investors’ opportunity set.

Another commonly used performance metric is funds’ CAPM alpha (Jensen (1968)). Let us consider the case where $\mu_Q^r = \mu_P^r$, and as a result, investors’ portfolio is proportional to the delegation portfolio (and the market portfolio):

$$\mathbf{w}^o = \left(\frac{1 - (\gamma + \theta) \left(\frac{\delta}{1-\delta} \right) \frac{2v}{\gamma}}{\gamma + \theta v} \right) (\Sigma_P^r)^{-1} (\mu_P^r - r_f \mathbf{1}). \quad (30)$$

Therefore, CAPM holds. A regression of funds returns on the market return shows exactly zero alpha in a large sample. After management fees, investors receive negative alpha from delegation. Moreover, managers hold the market portfolio up to a scaling factor, as have already been documented in the empirical literature (e.g., Lewellen (2011)).

Proposition 8 (Delegation and Alpha) *Under the simplified ambiguity, if $\mu_Q^r = \mu_P^r$, the delegation portfolio has zero alpha (negative after fees) and is proportional to the market.*

Another interesting implication of our model is that even if managers possess superior knowledge and know the true model, this may not help them to generate “market risk-adjusted return”. This result challenges the traditional approach of fund performance measurement: an asset management firm could be active in acquiring the knowledge of true return distribution, but such effort is unlikely to be compensated if we only look at alpha.

Simplified ambiguity premium. We derive several conditions on the relation between model uncertainty and assets’ CAPM alpha that directly guide our empirical analysis. Under

the three assumptions, the CAPM alpha in Equation (21) becomes

$$\boldsymbol{\alpha} = \underbrace{\left[\left(\frac{2(\theta/\gamma + 1)}{\theta/\gamma + 1/v} \right) \right]}_{\text{Delegation Hedging}} - \underbrace{\left[\left(\frac{1}{1 + v\theta/\gamma} \right) \left(\frac{1 - \delta}{\delta} \right) \right]}_{\text{Zero-Delegation Component}} \left(\mu_Q^r - r_f \mathbf{1} \right). \quad (31)$$

The cross-sectional variation of $\boldsymbol{\alpha}$ is from $\left(\mu_Q^r - r_f \mathbf{1} \right)$, the vector of expected excess returns under investors' average model. Therefore, $\boldsymbol{\alpha}$ dispersion becomes larger if its coefficient increases, for example when model uncertainty increases (higher v). In Section 3, we test whether the cross-section dispersion of CAPM alpha increases when uncertainty rises.

Proposition 9 (Uncertainty and Alpha Disperstion) *Given δ , the cross-section dispersion of alpha increases in model uncertainty, v , as shown by Equation (31).*

As δ approaches 100%, the component of CAPM alpha from the zero-delegation part of investors' portfolio shrinks to zero, while the component from delegation hedging remains. Interestingly, investors' average belief, which reflects potential behavioral biases, survives in $\boldsymbol{\alpha}$ through delegation hedging. This is not due to the limits to arbitrage in the existing models of behavioral finance (Barberis and Thaler (2003)). Here when investors delegate more, feeding managers with more arbitrage capital, they hedge more. Hedging against delegation uncertainty sustains alpha.¹⁸

Another application of our theory is to extract investors' model uncertainty from data. Measuring model uncertainty is very challenging because, by nature, ambiguity is subjective. However, as shown by Equation (31), if the model uncertainty, v , varies over time, the relation between alpha and investors' expectations varies. Therefore, if we could obtain a measure of investors' expectations, i.e., $\left(\mu_Q^r - r_f \mathbf{1} \right)$, we can back out the dynamics of investors' model uncertainty by projecting assets' CAPM residuals on investors' expectation in rolling windows while controlling for the delegation level. In Section 3, we estimate investors' model uncertainty by using surveys on investors' expectations as proxy for $\left(\mu_Q^r - r_f \mathbf{1} \right)$.

¹⁸Our model does not speak to the evolution of wealth distribution between rational and irrational agents over time. Goldman and Slezak (2003) study persistent erroneous information in asset prices from shorter tenures of fund managers than the time it takes for their private information to become public. Many studies focus on asset-price distortions due to managers' incentive problems, such as Basak and Pavlova (2013), Buffa, Vayanos, and Woolley (2014), Cuoco and Kaniel (2011), Dow and Gorton (1997), Guerrieri and Kondor (2012), Kaniel and Kondor (2013). Gârleanu and Pedersen (2018), Huang (2016), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), and Sockin and Zhang (2018) emphasize information acquisition.

Proposition 10 (Extracting Model Uncertainty from Data) *Conditional on the level of delegation, δ , the relation between assets’ CAPM alpha and investors’ expectations under the average model reveals the level of model uncertainty, as shown by Equation (31).*

3 Evidence

We provide evidence on our model assumptions and main results using data on asset returns, assets’ ownership by funds and individual investors in the U.S. stock market.

3.1 Data and Variable Construction

Our model is built upon the assumption that investors and managers have different beliefs on asset returns. Our main results, and in particular, the cross section of assets’ CAPM alpha, are determined by investors’ subjective model uncertainty. The challenge in testing our model assumption and results is that we do not observe investors’ and managers’ beliefs. Therefore, taking a revealed-preference approach, we examine their beliefs through the observed portfolio rebalancing and test the model predictions on asset returns.

Asset space. We use the well-studied equity factors instead of individual stocks as the asset universe, because these factors largely span individual stocks’ returns. Factors can be either accounting-based or return-based. Accounting-based factors include value (“HML”), accruals (“ACR”), investment (“CMA”), profitability (“RMW”), and net issuance (“NI”). Return-based factors include momentum (“MOM”), short-term reversal (“STR”), long-term reversal (“LTR”), betting-against-beta (“BAB”), idiosyncratic volatility (“IVOL”), and total volatility (“TVOL”). To construct each factor, we use monthly and daily returns data of stocks listed on NYSE, AMEX, and Nasdaq from the Center for Research in Securities Prices (CRSP).¹⁹ We obtain accounting data from annual COMPUSTAT files.²⁰

We construct each factor in the typical HML-like fashion by independently sorting stocks into six value-weighted portfolios by size (“ME”) and the factor characteristic. We use standard NYSE breakpoints – median for size, and 30th and 70th percentiles for the factor characteristic. A factor’s return is the value-weighted average of the two high-characteristic

¹⁹We include ordinary common shares (codes 10 and 11) and adjust delisting with CRSP delisting returns.

²⁰We follow the convention and lag accounting information by six months (Fama and French (1993)). If a firm’s fiscal year ends in Dec. of year t , we assume information is available at the end of Jun. of year $t + 1$.

portfolios minus that of the two low-characteristic portfolios. We rebalance accounting-based factors annually at the end of each June and rebalance the return-based factors monthly.

Portfolio allocation. To measure investors’ factor allocation, we use the Thomson-Reuters 13F Database from 1980Q1 to 2016Q4, which covers stock ownership by mutual funds, hedge funds, insurance companies, banks, trusts, pension funds, and other institutions. For each stock, we sum the institutional holdings and define the remaining fraction as individual ownership (“*INDV*”) following [Gompers and Metrick \(2001\)](#) and [Fang and Peress \(2009\)](#).

Ideally, we would like to treat each factor as an asset and compute the fraction owned by individual investors. However, factors are comprised of numerous stocks and different factors have overlapping constituents. For example, stock A could be in the long leg of value factor and the short leg of momentum factor. Instead of calculating the exact ownership, we calculate the relative over- or under-weight of each factor by individual investors. Specifically, we measure the spread of aggregate *INDV* between the factor’s long leg and short leg:

$$INDV_{i,t} = INDV_{i,t}^{long} - INDV_{i,t}^{short} \quad (32)$$

where $INDV_{i,t}^j$, $j \in \{long, short\}$, is the value-weighted average of the individual ownership of all constituent stocks in j leg of factor i .

We follow a similar approach to measure managers’ factor ownership, but only use data on mutual funds because their investment objective code (IOC) can be obtained from CRSP. We select funds focusing on the U.S. stock market using IOC, excluding International, Municipal Bonds, Bond & Preferred, and Balanced.²¹ We aggregate fund holdings for each stock. Stocks are assumed to have zero fund ownership if they appear in CRSP but without any reported fund holdings. We calculate the relative over- or under-weight of each factor by managers as the spread of fund ownership (“*INST*”) between the long leg and short leg:

$$INST_{i,t} = INST_{i,t}^{long} - INST_{i,t}^{short} \quad (33)$$

where $INST_{i,t}^j$, $j \in \{long, short\}$, is the value-weighted average of fund ownership of all

²¹We apply standard filters following the literature: (1) we pick the first vintage date (“*FDATE*”) for each fund-report date (*FUNDNO-RDATE*) pair to avoid stale information; (2) we adjust shares held by a fund for stock splits that happen between report date (“*RDATE*”) and vintage date (“*FDATE*”). As a robustness check, we select only active domestic equity funds, and find similar results (available upon request).

constituent stocks in j leg of factor i . If managers find factor i 's return distribution is desirable, they increase exposure to i , and accordingly, $INST_{i,t}$ increases.

Uncertainty measures. We obtain measures of uncertainty in the existing literature to test our model predictions and to compare with our model-implied measure of uncertainty. In particular, we consider the first principal component (\mathcal{U}_t^{PCA}) and cross-sectional average (\mathcal{U}_t^{CSA}) of uncertainties estimated using macro and financial variables from [Jurado, Ludvigson, and Ng \(2015\)](#); Economic Policy Uncertainty (EPU) and news-based Economic Policy Uncertainty (EPU^{news}) from [Baker, Bloom, and Davis \(2016\)](#); CBOE stock market volatility indexes VIX and VXO ([Williams \(2015\)](#)). We include volatility measures because, as shown in Section 2.5, volatilities affect the uncertainty (estimation errors) in the mean.²²

Investor expectations. To extract investors' ambiguity from data, we need investors' expectations on asset returns under the average model, \bar{Q} . We use the survey forecasts from the American Association of Individual Investors Sentiment Survey ("AA"), which measures the percentage of individual investors who are bullish, neutral, or bearish on the stock market for the next six months.²³ Following [Greenwood and Shleifer \(2014\)](#), we construct a time series of investor expectations by subtracting the percentage of bearish investors from the percentage of bullish investors, and average the weekly data to monthly frequency.

Table 1 reports summary statistics of factor returns, the factors' fund and investor ownership, the uncertainty measures, and the survey expectations.

[Insert Table 1 here.]

3.2 The Cross Section of Factor Alpha

We test our main results on asset pricing, Proposition 5, and characterize the cross section of factors' CAPM alpha. The challenge is that as shown in Equation (21), alphas (ambiguity premia) depend on investors' subjective belief, i.e., the set of candidate models, Δ , and their

²²Related, [Driouchi, Trigeorgis, and So \(2018\)](#) introduce ambiguity into the Black-Scholes model through Choquet-Brownian motions, and measure ambiguity by minimizing the absolute error between index options prices and the model-implied values. [Andreouyz, Kagkadix, Maio, and Philipz \(2014\)](#) measure ambiguity using the dispersion of index option strike prices. [Ulrich \(2013\)](#) uses inflation entropy to measure ambiguity.

²³[Greenwood and Shleifer \(2014\)](#) show that this qualitative measure captures similar dynamics as the quantitative measures from surveys that explicitly ask individuals' numeric expectations of market returns.

prior, π , which cannot be estimated from data. To address this issue, we back out investors' belief from the observed portfolio rebalancing, taking a revealed-preference approach.

Consider an increase of model uncertainty. Investors' hedging against delegation uncertainty becomes stronger. They increase positions in assets whose expected returns move against that of the efficient frontier across models (delegation-hedging assets), while decrease positions in assets whose expected returns comove with that of the efficient frontier. We rank factors by the correlation between individual ownership ($INDV$) and uncertainty. The model predicts smaller CAPM alphas of assets with higher correlations, which, revealed by investors' portfolio rebalancing, offer better insurance against delegation uncertainty.

[Insert Table 2 here.]

Table 2 confirms the prediction. We divide the eleven factors into equal-weighted high ("H") portfolio (six factors) and low ("L") portfolio (five factors) by the correlation between a factor's individual ownership and the uncertainty measure, \mathcal{U}_t^{PCA} , from Jurado, Ludvigson, and Ng (2015). The annualized CAPM alphas are reported together with t-statistic. The alpha of H portfolio is indistinguishable from zero. It contains delegation-hedging assets as investors overweigh these assets as uncertainty rises. The L portfolio, which exposes investors to more delegation uncertainty, carries a CAPM alpha of 2.67%. The L-minus-H portfolio has a significant CAPM alpha of 2.59%. In the right panel, we estimate CAPM alphas after controlling for the delegation level, which maps more closely to Equation (21). Table A.1 in the appendix shows a similar pattern when other uncertainty measures are used.

[Insert Figure 1 and 2 here.]

Figure 1 plots the CAPM alphas of H and L portfolios estimated in 60-month rolling windows. Except for a period in the early 2000s, the alpha of H portfolio dominates that of L portfolio. This suggests that the results in Table 2 are not driven by a particular episode.

In Panel A of Figure 2, we plot the full-sample alphas of all factors against the correlation between their individual ownership and uncertainty, and in Panel B, we control for δ . The model predicts a negative relation in the cross section (a downward-sloping regression line), which largely holds in data except for the momentum factor. For comparison, we plot the cross section without momentum in Panel C and D.

3.3 Uncertainty and Time-Varying Alpha Dispersion

Having characterized the cross section of factor alphas, we next test our model’s prediction on how the cross section varies over time. Proposition 9 states that given δ , the cross-section dispersion of alpha increases in the level of model uncertainty (Equation (31)).

[Insert Figure 3 here.]

Figure 3 plots for each month, the cross-section dispersion (the difference between maximum and minimum) of factors’ CAPM residuals against each of the six uncertainty measures. The uncertainty measures are lagged by a month because the model implies a relation between uncertainty and the *expected* dispersion of CAPM residuals (i.e., the alpha dispersion) rather than the realized dispersion. A strong positive correlation emerges. Figure A.1 in the appendix reports similar patterns for the dispersion of factors’ raw returns.

[Insert Table 3 here.]

Table 3 reports the results of parametric tests. We consider two measures of dispersion, the difference between max and min, and the cross-section standard deviation of factors’ CAPM residuals. We forecast the dispersion with uncertainty measures (Panel A). In Panel B and C, we control for the raw and detrended aggregate fund ownership (i.e., δ in data) respectively, mapping the specifications more closely to Equation (31). Table A.2 in the appendix reports the results for factor return dispersion.

Across specifications, measures of uncertainty positively predict the dispersion of factors’ CAPM residuals and returns. The economic magnitude is sizable. For example, an 1% (one standard deviation) increase of \mathcal{U}_t^{CSA} predicts a 0.46% (annualized to 5.52%) increase of the cross-section standard deviation of factors’ CAPM residuals.

3.4 Factor Timing by Fund Managers

We test our modeling assumption that managers know the asset return distribution better than investors. Here we also take a revealed-preference approach. As we have shown in Section 3.3, the cross section of factor alphas and returns vary over time. If our assumption holds in data, we should be able to observe that fund managers rebalance their portfolio towards factors with superior distributional properties in the next period.

Specifically, we estimate the following predictive regression: for factor i at time t ,

$$R_{i,t,t+3} = \alpha + \beta \cdot INST_{i,t} + \gamma \cdot X_{i,t} + \varepsilon_{i,t,t+3} \quad (34)$$

where $i = \{HML, ACR, CMA, RMW, NI, MOM, STR, LTR, BAB, IVOL, TVOL\}$, and $R_{i,t,t+3}$ is the return next quarter (i.e., month t to $t + 3$), and $X_{i,t}$ includes control variables such as factor volatility that may also predict factor returns (Moreira and Muir (2017)). We use the next-quarter return because institutional ownership data is available quarterly for individual stocks. Note that $INST$ at factor level varies every month due to the monthly rebalancing of value-weighted factor portfolios. Therefore, our estimation is at monthly level but with overlapping left-hand side variables. Our hypothesis is that a factor will deliver higher return in the future if its manager ownership, $INST$, increases now.

[Insert Table 4 here.]

To increase statistical power, we pool factors together to a panel predictive regression. In Table 4 Panel A, we report the results using pooled OLS and various fixed effects. $RV_{i,t}$ is the realized volatility of factor i estimated using previous 36 months of returns (Moreira and Muir (2017)). Standard errors are double-clustered by factor and quarter.

As typical in the literature of return predictability, we address the concern over biased standard errors due to overlapping observations. We follow the suggestion of Hodrick (1992) and run the following “reverse” regression to test the return predictability:

$$3 \times R_{i,t+1} = \alpha + \beta \left(\frac{1}{3} \sum_{j=0}^2 INST_{i,t-j} \right) + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1}. \quad (35)$$

On the left-hand side is $R_{i,t+1m}$, the future one-month return multiplied by 3 so that it is comparable in magnitude with quarterly returns. Results are reported in Table 4 Panel B.

Our modeling assumption is confirmed in all specifications. In both panels, the predictive coefficient of $INST$ is positive and significant, robust to alternative standard errors and various fixed effects. The coefficients in simple predictive regressions and the Hodrick reverse regressions are very close. Moreover, the predictability is economically meaningful. For example, the coefficient 0.31 in the first column of Panel B implies that, when $INST$ of a factor rises by one standard deviation, the return increases by 44 bps in the following

quarter (1.76% annualized). Given the average annual factor return of 3.31% in our sample, this is a 53% increase over the average. The evidence of factor timing by fund managers lends support to our assumption that managers possess superior knowledge of return distribution.

[Insert Figure 4 here.]

As a non-parametric test, we rank factors by their *INST* at the end of each quarter, and form equal-weighted high (four factors), medium (three factors), and low (four factors) portfolios. As shown in Panel A of Figure 4, high-*INST* factors consistently outperform low-*INST* ones since 1991. The fact that this pattern started in the early 1990s suggests that asset managers may have benefited from the exploding research efforts devoted to equity factors, more data sources, and the developments of financial econometrics before the 1990s.

Another prediction of our model is that the asset-market equilibrium does not converge to CAPM as the level of delegation rises. To examine this property, we plot the CAPM alpha of the high-*INST* portfolio (left Y-axis) and the aggregate fund ownership (right Y-axis), i.e. δ , in Panel B of Figure 4. While the latter has trended up in the recent decades, the former also increased with occasional decline. Overall there is no evidence that a growing asset management sector is associated with declining alpha and convergence to CAPM.

[Insert Table 5 here.]

So far, we have examined the first moment of factor returns. In Table 5, we report higher moments and other statistics of factor portfolio returns. High *INST* factors exhibit high mean return, low volatility, and small skewness. These statistics vary monotonically in *INST*, suggesting that asset managers tend to invest in factors with a desirable statistical profile. Managers also tend to hold stocks with higher kurtosis. Under ambiguity, investors refrain from factors with more extreme returns, while asset managers are more willing to take on such exposure possibly due to their confidence in gauging the return distribution.

3.5 Model-Implied Uncertainty Measure

Due to the subjective nature of ambiguity, it is challenging to measure the model uncertainty that investors face when making delegation and asset-allocation decisions. Proposition 10 shows how to extract ambiguity from assets' CAPM alpha, investors' expectations, and delegation. Next we use a two-step procedure to estimate the model-implied uncertainty, v .

First, given a 60-month window starting in month t , we run a panel regression of factors' excess returns on the market excess return and survey expectations: for factor i in month $s \in [t, t + 59]$,

$$r_{i,s} - r_{f,s} = a_t + b_{i,t} \times (r_{M,s} - r_{f,s}) + c_t \times survey_{s-1} + \varepsilon_{i,s}, \quad (36)$$

where $r_{f,s}$ is the risk-free rate and the coefficients' subscript t marks the rolling window.

This regression is the empirical counterpart of Equation (31). However, the left-hand side of Equation (31) is CAPM alpha, while that of the regression is realized factor return. Therefore, we control for the market excess return and allow different factors to have different betas (i.e., control for the whole CAPM component). Moreover, *survey* is lagged because in the model, investors' expectations are matched with ex ante alpha instead of ex post, realized CAPM residuals. Finally, note that our survey data is on investors' expectations of future *market* return instead of individual factors' returns, i.e., $(\mu_Q^r - r_f \mathbf{1})$. It is an imperfect proxy, but readily available and one of the most widely used survey variables.

Next, we use the time series of regression coefficient \hat{c}_t to back out investors' model uncertainty. In the model, \hat{c}_t combines both the level of delegation δ and the model uncertainty v . Therefore, we regress \hat{c}_t on the time-t aggregate fund ownership, δ_t , and take the OLS residual as our model-implied measure of investors' ambiguity, which we denote by \hat{v}_t .

[Insert Figure 5 here.]

Figure 5 plots the time series of our estimated model uncertainty \hat{v}_t , together with the composite uncertainty measure (\mathcal{U}^{PCA}) extracted from a large set of macro and financial variables by Jurado, Ludvigson, and Ng (2015), Economic Policy Uncertainty (*EPU*) of Baker, Bloom, and Davis (2016), and CBOE stock market volatility index (*VIX*). The estimated model uncertainty exhibits an economically meaningful dynamics, peaking around major episodes of market turmoils such as the dotcom bubble and the financial crisis. It carries information distinct from other uncertainty measures. Even though different measures are not capturing the same object in theory, they are correlated. Specifically, our uncertainty measure, \hat{v}_t , has a correlation of 0.5 with \mathcal{U}^{PCA} .

4 Conclusion

A division of knowledge between managers and investors leads to delegation, but at the same time, generates delegation uncertainty. We highlight the welfare gains from delegation that resolve several puzzles on delegated portfolio management. Our theory also delivers asset pricing implications supported by evidence. A key insight is that investors' hedging against delegation uncertainty creates CAPM alpha that is immune to the rise of arbitrage capital.

Delegation uncertainty arises wherever agents differ in their access to information. While we focus on the application in financial markets, similar research questions can be cast in other economic settings. For example, communication within an organization is imperfect given the scarce attention ([Dessein, Galeotti, and Santos \(2016\)](#)). In such cases, delegation uncertainty induces distortion in the resource allocation within an organization.

Ambiguity has attracted enormous attention in the macroeconomics literature ([Bianchi, Ilut, and Schneider \(2018\)](#)). Informational specialization and delegation are ubiquitous, but are often ignored in macroeconomic models. By showing that delegation significantly changes the role of uncertainty in agents' decision making, our work suggests that incorporating delegation can bring new insights on the macroeconomic consequences of model uncertainty.

Appendix I: Quadratic Approximation

Define q as the Radon-Nikodym derivative of Q w.r.t. \bar{Q} , i.e., $q(\omega) = \frac{dQ(\omega)}{d\bar{Q}(\omega)}$ for $\omega \in \Omega$. In the following, we use Q and q interchangeably to denote a candidate probability model. Define the following function corresponding to the certainty equivalent:

$$F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d) = C \left(r_f + (\mathbf{r} - r_f \mathbf{1})^T [(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d(q)] \right)$$

Hence, $F : B(\Omega, \mathbf{R}) \times \mathbf{R}^N \times L^\infty \mapsto \mathbf{R}$ is a functional defined on three Banach spaces, where $B(\Omega, \mathbf{R})$ denotes the set of mappings from Ω to \mathbf{R} .

Frechet derivatives of C . Here we list several useful expressions and definitions

- $(v^{-1}(\cdot))' = \frac{1}{v'(v^{-1}(\cdot))}$ and $\phi'(\cdot) = (v \circ u^{-1}(\cdot))' = \frac{v'(u^{-1}(\cdot))}{u'(u^{-1}(\cdot))}$.
- $(v^{-1}(\cdot))'' = -\frac{1}{[v'(v^{-1}(\cdot))]^2} \frac{v''(v^{-1}(\cdot))}{v'(v^{-1}(\cdot))}$.
- $\phi''(\cdot) = (v \circ u^{-1}(\cdot))'' = \frac{v'(u^{-1}(\cdot))}{[u'(u^{-1}(\cdot))]^2} \left[\frac{v''(u^{-1}(\cdot))}{v'(u^{-1}(\cdot))} - \frac{u''(u^{-1}(\cdot))}{u'(u^{-1}(\cdot))} \right]$.
- Define $\gamma = -\frac{u''(r_f)}{u'(r_f)}$ and $\theta = -u'(r_f) \frac{\phi''(u(r_f))}{\phi'(u(r_f))} = -\left[\frac{v''(r_f)}{v'(r_f)} - \frac{u''(r_f)}{u'(r_f)} \right]$.
- Denote $D_{\mathbf{w}^o} F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$ and $D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$ to be the first-order Fréchet derivatives of C with respect to \mathbf{w}^o and \mathbf{w}^d , and $D_{\mathbf{w}^o}^2 F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$ and $D_{\mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$ to be the second-order Fréchet derivatives of C with respect to \mathbf{w}^o and \mathbf{w}^d .
- Denote $V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d) = \int_{\Delta} \phi \left(\int_{\Omega} u(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) dQ(\omega) \right) d\pi(q)$, so $V(\mathbf{r}, \mathbf{0}, \mathbf{0}) = \phi(u(r_f))$.
- Denote $U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)) = \int_{\Omega} u(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) dQ(\omega)$, so $U(\mathbf{r}, \mathbf{0}, \mathbf{0}) = u(r_f)$.
- For any random variable R and probability measure P , μ_P^R denotes the mean of R under P , Σ_P^R the covariance of R under P if R is vector and $\sigma_P^2(R)$ the variance under P if R is scalar.

Derivatives w.r.t. \mathbf{w}^d . First, calculate the Fréchet derivatives of $V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$

$$\begin{aligned}
& D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}) \\
&= \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \frac{\partial U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))}{\partial \mathbf{w}^d(q)} \boldsymbol{\delta}(q) d\pi(q) \\
&= \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) \delta(\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}(q) dQ(\omega) d\pi(q)
\end{aligned}$$

which is a row vector, and

$$\begin{aligned}
& D_{\mathbf{w}^d}^2 V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \\
&= \int_{\Delta} \phi''(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \left(\int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) \delta(\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_2(q) dQ(\omega) \right) \\
&\quad \left(\int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) \delta(\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_1(q) dQ(\omega) \right) d\pi(q) + \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \\
&\quad \int_{\Omega} u''(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) \delta^2 \boldsymbol{\delta}_1(q)^T (\mathbf{r} - r_f \mathbf{1}) (\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_2(q) dQ(\omega) d\pi(q)
\end{aligned}$$

which is a N -by- N matrix. Evaluate at $(\mathbf{w}^o, \mathbf{w}^d) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = \mathbf{w}^d$:

$$\begin{aligned}
D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^d) &= v'(r_f) \delta E_{\pi} \left(E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right) \\
D_{\mathbf{w}^d}^2 V(\mathbf{r}, \mathbf{0}, \mathbf{0}) &= \phi''(u(r_f)) [u'(r_f)]^2 \delta^2 E_{\pi} \left(\left[E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right]^2 \right) + \\
&\quad \phi'(u(r_f)) u''(r_f) (\delta^2) E_{\pi} \left(E_Q \left(\left[(\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right]^2 \right) \right)
\end{aligned}$$

By chain rule,

$$\begin{aligned}
D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}) &= \frac{D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta})}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))} \\
&= \int_{\Delta} \frac{\phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)))}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))} \int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}) \delta(\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}(q) dQ(\omega) d\pi(q) \\
D_{\mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) &= - \frac{v''(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))}{[v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))]^3} [D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1)] \\
&\quad [D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_2)] + \frac{D_{\mathbf{w}^d}^2 V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))}
\end{aligned}$$

Evaluate at $(\mathbf{w}^o, \mathbf{w}^d) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = \mathbf{w}^d$:

$$\begin{aligned} D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^d) &= \delta E_\pi \left(E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right) \\ D_{\mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^d, \mathbf{w}^d) &= -\theta \delta^2 \text{Var}_\pi \left(E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right) - \\ &\quad \gamma \delta^2 E_\pi \left(\sigma_Q^2 \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right) \end{aligned}$$

Derivatives w.r.t. \mathbf{w}^o . First, calculate the Fréchet derivatives of $V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)$:

$$\begin{aligned} &D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}) \\ &= \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \frac{\partial U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))}{\partial \mathbf{w}^o} \boldsymbol{\delta} d\pi(q) \\ &= \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d})(1 - \delta) (\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta} dQ(\omega) d\pi(q) \end{aligned}$$

which is a row vector, and

$$\begin{aligned} &D_{\mathbf{w}^o}^2 V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \\ &= \int_{\Delta} \phi''(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \left(\int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d})(1 - \delta) (\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_1 dQ(\omega) \right) \\ &\quad \left(\int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d})(1 - \delta) (\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_2 dQ(\omega) \right) d\pi(q) + \\ &\quad \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \int_{\Omega} u''(r_{\delta, \mathbf{w}^o, \mathbf{w}^d})(1 - \delta)^2 \\ &\quad \boldsymbol{\delta}_1^T (\mathbf{r} - r_f \mathbf{1}) (\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta}_2 dQ(\omega) d\pi(q) \end{aligned}$$

which is a N -by- N matrix. Evaluate at $(\mathbf{w}^o, \mathbf{w}^d) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = \mathbf{w}^o$:

$$\begin{aligned} D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o) &= (1 - \delta) v'(r_f) E_{\bar{Q}} \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right) \\ D_{\mathbf{w}^o}^2 V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o, \mathbf{w}^o) &= \phi''(u(r_f)) [u'(r_f)]^2 (1 - \delta)^2 E_\pi \left(\left[E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right) \right]^2 \right) \\ &\quad + \phi'(u(r_f)) u''(r_f) (1 - \delta)^2 E_{\bar{Q}} \left(\left[(\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right]^2 \right) \end{aligned}$$

Then,

$$\begin{aligned}
D_{\mathbf{w}^o} F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}) &= \frac{D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta})}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))} \\
&= \frac{1}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))} \int_{\Delta} \phi'(U(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q))) \\
&\quad \int_{\Omega} u'(r_{\delta, \mathbf{w}^o, \mathbf{w}^d})(1-\delta)(\mathbf{r} - r_f \mathbf{1})^T \boldsymbol{\delta} dQ(\omega) d\pi(q)
\end{aligned}$$

and

$$\begin{aligned}
&D_{\mathbf{w}^o}^2 F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \\
= &-\frac{v''(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))}{[v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))]^3} [D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1)] [D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_2)] \\
&+ \frac{D_{\mathbf{w}^o}^2 V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\mathbf{w}^o, \mathbf{w}^o)}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))}
\end{aligned}$$

Evaluate at $(\mathbf{w}^o, \mathbf{w}^d) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = \mathbf{w}^o$:

$$\begin{aligned}
D_{\mathbf{w}^o} F(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o) &= (1-\delta) \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right)^T \mathbf{w}^o \\
D_{\mathbf{w}^o}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o, \mathbf{w}^o) &= -\theta (1-\delta)^2 \text{Var}_{\pi} \left(E_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right) \right) - \\
&\quad \gamma (1-\delta)^2 \text{Var}_{\overline{Q}} \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o \right)
\end{aligned}$$

Second derivatives w.r.t. \mathbf{w}^d and \mathbf{w}^o . Finally,

$$\begin{aligned}
&D_{\mathbf{w}^o \mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \\
= &\frac{D_{\mathbf{w}^o \mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))} - \frac{[v''(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))/v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))]}{[v'(v^{-1}(V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)))]^2} \\
&[D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_1)] [D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d)(\boldsymbol{\delta}_2)]
\end{aligned}$$

Evaluate at $(\mathbf{w}^o, \mathbf{w}^d) = \mathbf{0}$ and $\boldsymbol{\delta}_1 = \mathbf{w}^o, \boldsymbol{\delta}_2 = \mathbf{w}^d$:

$$\begin{aligned}
&D_{\mathbf{w}^o \mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o, \mathbf{w}^d) \\
= &\frac{D_{\mathbf{w}^o \mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o, \mathbf{w}^d)}{v'(r_f)} - \frac{v''(r_f)}{[v'(r_f)]^3} [D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^o)] [D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0})(\mathbf{w}^d)]
\end{aligned}$$

where

$$\begin{aligned}
& D_{\mathbf{w}^o \mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o, \mathbf{w}^d) \\
&= -v'(r_f) \theta (1 - \delta) \delta \int_{\Delta} \mathbf{w}^d(q)^T E_Q(\mathbf{r} - r_f \mathbf{1}) E_Q\left((\mathbf{r} - r_f \mathbf{1})^T\right) \mathbf{w}^o d\pi(q) \\
&\quad - v'(r_f) \gamma (1 - \delta) \delta \int_{\Delta} \mathbf{w}^{oT} E_Q\left((\mathbf{r} - r_f \mathbf{1})(\mathbf{r} - r_f \mathbf{1})^T\right) \mathbf{w}^d(q) d\pi(q)
\end{aligned}$$

Simplify the expression:

$$\begin{aligned}
& D_{\mathbf{w}^o \mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o, \mathbf{w}^d) \\
&= \frac{D_{\mathbf{w}^o \mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o, \mathbf{w}^d)}{v'(r_f)} - \frac{v''(r_f)}{[v'(r_f)]^3} [D_{\mathbf{w}^o} V(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o)] [D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^d)] \\
&= -(\theta + \gamma) (1 - \delta) \delta \text{cov}_{\pi} \left(E_Q\left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o\right), E_Q\left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q)\right) \right) \\
&\quad - \gamma (1 - \delta) \delta E_{\pi} \left(\text{cov}_Q \left((\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^o, (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}^d(q) \right) \right)
\end{aligned}$$

Taylor expansion of C . By Theorem 8.16 of [Jost \(2005\)](#),

$$\begin{aligned}
& C\left(r_f + (\mathbf{r} - r_f \mathbf{1})^T [(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d(q)]\right) = F(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d) \\
&= r_f + D_{\mathbf{w}^o} F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o) + D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^d) \\
&\quad + \frac{1}{2} D_{\mathbf{w}^o}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o, \mathbf{w}^o) + \frac{1}{2} D_{\mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^d, \mathbf{w}^d) \\
&\quad + D_{\mathbf{w}^o \mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0}) (\mathbf{w}^o, \mathbf{w}^d) + R(\mathbf{w}^o, \mathbf{w}^d)
\end{aligned}$$

where $D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{0}, \mathbf{0})$, $D_{\mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})$, $D_{\mathbf{w}^o} F(\mathbf{r}, \mathbf{0}, \mathbf{0})$, $D_{\mathbf{w}^o}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})$, and $D_{\mathbf{w}^o \mathbf{w}^d}^2 F(\mathbf{r}, \mathbf{0}, \mathbf{0})$ have been solved and $\lim_{(\mathbf{w}^o, \mathbf{w}^d) \rightarrow \mathbf{0}} \frac{R(\mathbf{w}^o, \mathbf{w}^d)}{\|(\mathbf{w}^o, \mathbf{w}^d)\|^2} = 0$, where the denominator is the L^2 norm.

To simplify the notations, let $R^{\mathbf{w}}$ denote the *excess* return generated by any portfolio \mathbf{w} and let $R_P^{\mathbf{w}}$ denotes the expected *excess* return of any portfolio \mathbf{w} under probability measure P . Also notice that $\mathbf{w}^d(q) = (\gamma \Sigma_Q^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})$. We have:

$$E_{\pi} \left(\text{cov}_Q \left(R^{\mathbf{w}^o}, R^{\mathbf{w}^d} \right) \right) = E_{\pi} \left(\mathbf{w}^{oT} \Sigma_Q^{\mathbf{r}} \mathbf{w}^d(q) \right) = \frac{1}{\gamma} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right)^T \mathbf{w}^o = \frac{1}{\gamma} R_Q^{\mathbf{w}^o}$$

Taylor expansion can be simplified as

$$\begin{aligned}
& C \left(r_f + (\mathbf{r} - r_f \mathbf{1})^T [(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d(q)] \right) \\
& \approx r_f + (1 - \delta) R_Q^{\mathbf{w}^o} + \delta E_\pi \left(R_Q^{\mathbf{w}^d} \right) - (\theta + \gamma) (1 - \delta) \delta \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) \\
& \quad - \frac{(1 - \delta)^2}{2} \left(\gamma \sigma_Q^2 \left(R^{\mathbf{w}^o} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^o} \right) \right) - \frac{\delta^2}{2} \left(\gamma E_\pi \left(\sigma_Q^2 \left(R^{\mathbf{w}^d} \right) \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) \right)
\end{aligned}$$

Appendix II: Optimal Portfolio and Delegation

The investor's problem is

$$\max_{\mathbf{w}^o, \delta} C \left(r_{\delta, \mathbf{w}^o, \mathbf{w}^d} \right) - \delta \psi$$

given that

$$\mathbf{w}^d(q) = (\gamma \Sigma_Q^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})$$

Approximate $C \left(r_{\delta, \mathbf{w}^o, \mathbf{w}^d} \right)$:

$$\begin{aligned}
& C \left(r_f + (\mathbf{r} - r_f \mathbf{1})^T [(1 - \delta) \mathbf{w}^o + \delta \mathbf{w}^d] \right) \\
& \approx r_f + (1 - \delta)^2 \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right)^T \mathbf{w}^o - (\theta + \gamma) (1 - \delta) \delta \text{cov}_\pi \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}, R_Q^{\mathbf{w}^d} \right)^T \mathbf{w}^o \\
& \quad - \frac{(1 - \delta)^2}{2} \left(\gamma \mathbf{w}^{oT} \Sigma_Q^{\mathbf{r}} \mathbf{w}^o + \theta \mathbf{w}^{oT} \Sigma_\pi^{\mu_Q^{\mathbf{r}}} \mathbf{w}^o \right) + \delta E_\pi \left(R_Q^{\mathbf{w}^d} \right) \\
& \quad - \frac{\delta^2}{2} \left(\gamma E_\pi \left(\sigma_Q^2 \left(R^{\mathbf{w}^d} \right) \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) \right)
\end{aligned}$$

The first order condition of \mathbf{w}^o :

$$\mathbf{w}^o = \left(\gamma \Sigma_Q^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}} \right)^{-1} \left[\left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) - (\theta + \gamma) \frac{\delta}{1 - \delta} \text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right) \right]$$

From the first order condition of δ , δ equal to

$$\frac{\gamma \sigma_Q^2 \left(R^{\mathbf{w}^o} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^o} \right) + E_\pi \left(R_Q^{\mathbf{w}^d} \right) - 2 R_Q^{\mathbf{w}^o} - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) - \psi}{\gamma \sigma_Q^2 \left(R^{\mathbf{w}^o} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^o} \right) + E_\pi \left(R_Q^{\mathbf{w}^d} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) - 2 R_Q^{\mathbf{w}^o} - 2 (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right)},$$

where $\gamma\sigma_Q^2(R^{\mathbf{w}^o}) + \theta\sigma_\pi^2(R_Q^{\mathbf{w}^o})$ can be simplified because

$$\begin{aligned} & \mathbf{w}^{oT} \left[\left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right) - (\theta + \gamma) \frac{\delta}{1 - \delta} \text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right) \right] \\ &= R_Q^{\mathbf{w}^o} - (\theta + \gamma) \frac{\delta}{1 - \delta} \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) \end{aligned}$$

So,

$$1 - \delta = \frac{\theta\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) + \psi}{E_\pi \left(R_Q^{\mathbf{w}^d} \right) + \theta\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) - R_Q^{\mathbf{w}^o} - \left(\frac{2-\delta}{1-\delta} \right) (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right)}$$

Divide both sides by $1 - \delta$ and rearrange: δ is equal to

$$\frac{E_\pi \left(R_Q^{\mathbf{w}^d} \right) - R_Q^{\mathbf{w}^o} - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right) - \psi}{E_\pi \left(R_Q^{\mathbf{w}^d} \right) + \theta\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) - R_Q^{\mathbf{w}^o} - (\theta + \gamma) \text{cov}_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right)}$$

Next, we substitute the investor's optimal portfolio into δ to solve the optimal delegation level as a function of exogenous parameters.

Corollary 3 (Optimal delegation) *Proposition 12 and 3 jointly solve the investor's optimal delegation level δ :*

$$\delta = \frac{E_\pi \left(R_Q^{\mathbf{w}^d} \right) - (\theta + \gamma) B - C - \psi}{E_\pi \left(R_Q^{\mathbf{w}^d} \right) + \theta\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) - (\theta + \gamma)^2 A - 2(\theta + \gamma) B - C}, \quad (37)$$

where

$$A = \text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right)^T \left(\gamma \Sigma_Q^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}} \right)^{-1} \text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right), \quad (38)$$

$$B = \text{cov}_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d} \right)^T \left(\gamma \Sigma_Q^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}} \right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right), \quad (39)$$

$$C = \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right)^T \left(\gamma \Sigma_Q^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}} \right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right). \quad (40)$$

The solution in Equation (37) depends on the structure of the investor's model uncertainty that involves the cross-model mean and variance of expected delegation return and the cross-model comovement of delegation return and asset returns.²⁴ In Appendix IV, we show how to calibrate our model with real data and calculate the model-implied delegation.

²⁴To solve δ , we substitute the investor's optimal portfolio into Equation (12), so the formula is solved

Appendix III: Analysis under the Simplified Ambiguity

Optimal delegation and portfolio. First, we rewrite $cov_\pi(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d})$ under the three assumptions that simplify the structure of model uncertainty. The expected delegation return under probability model Q is

$$R_Q^{\mathbf{w}^d} = (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T \mathbf{w}^d(Q) = \frac{1}{\gamma} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})$$

We can rewrite and decompose the cross-model covariance between the expected asset returns and the expected delegation return as follows.

$$\begin{aligned} cov_\pi(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}) &= cov_\pi(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}) \\ &= \frac{1}{\gamma} cov_\pi\left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}\right)^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})\right) \\ &\quad + \frac{1}{\gamma} cov_\pi\left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}\right)^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})\right) \\ &\quad + \frac{1}{\gamma} cov_\pi\left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - r_f \mathbf{1})\right) \end{aligned}$$

To proceed, first, we recognize that $(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})$ is a linear combination of $(\mu_Q^{\mathbf{r}_i} - \mu_Q^{\mathbf{r}_i}) (\mu_Q^{\mathbf{r}_j} - \mu_Q^{\mathbf{r}_j})$ weighted by the elements of $(\Sigma_P^{\mathbf{r}})^{-1}$. Under the assumption that π is Gaussian, we use Isserlis' theorem to eliminate the first term. For any asset k ,

$$\begin{aligned} & cov_\pi\left(\mu_Q^{\mathbf{r}_k} - \mu_Q^{\mathbf{r}_k}, \left(\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}}\right)^T (\Sigma_P^{\mathbf{r}})^{-1} (\mu_Q^{\mathbf{r}} - \mu_Q^{\mathbf{r}})\right) \\ &= \sum_{i,j} (\Sigma_P^{\mathbf{r}})^{-1}_{(i,j)} \left(E_\pi \left[\left(\mu_Q^{\mathbf{r}_k} - \mu_Q^{\mathbf{r}_k}\right) \left(\mu_Q^{\mathbf{r}_i} - \mu_Q^{\mathbf{r}_i}\right) \left(\mu_Q^{\mathbf{r}_j} - \mu_Q^{\mathbf{r}_j}\right) \right] \right. \\ &\quad \left. - E_\pi \left(\mu_Q^{\mathbf{r}_k} - \mu_Q^{\mathbf{r}_k}\right) E \left[\left(\mu_Q^{\mathbf{r}_i} - \mu_Q^{\mathbf{r}_i}\right) \left(\mu_Q^{\mathbf{r}_j} - \mu_Q^{\mathbf{r}_j}\right) \right] \right) = 0, \end{aligned}$$

because first, $E_\pi \left[\left(\mu_Q^{\mathbf{r}_k} - \mu_Q^{\mathbf{r}_k}\right) \left(\mu_Q^{\mathbf{r}_i} - \mu_Q^{\mathbf{r}_i}\right) \left(\mu_Q^{\mathbf{r}_j} - \mu_Q^{\mathbf{r}_j}\right) \right]$ is the expectation of three zero-mean normal random variables, and thus, is equal zero, and second, $E_\pi \left(\mu_Q^{\mathbf{r}_k} - \mu_Q^{\mathbf{r}_k}\right) = E_\pi \left(\mu_Q^{\mathbf{r}_k}\right) -$

under the assumption of an interior solution, i.e., $\delta < 1$. When $\delta = 1$ and the investor does not retain any wealth to manage on her own, the investor's optimal portfolio given by Equation (13) is not well defined. This explains why even if delegation is free (i.e., $\psi = 0$), Equation (37) does not give 100% delegation. Intuitively, since the manager forms the efficient portfolio under each probability model, the investor with quadratic utility should fully delegate when $\psi = 0$. Therefore, the complete solution of delegation should be 100% if $\psi = 0$, and the interior value given by Equation (37) if $\psi > 0$.

$$\mu_{\bar{Q}}^{\mathbf{r}k} = \mu_{\bar{Q}}^{\mathbf{r}k} - \mu_{\bar{Q}}^{\mathbf{r}k} = 0.$$

Therefore, we have

$$\begin{aligned} & cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_{\bar{Q}}^{\mathbf{w}^d} \right) \\ &= \frac{1}{\gamma} cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}, \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}} \right)^T (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \right) \\ & \quad + \frac{1}{\gamma} cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}, \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right)^T (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \right). \end{aligned}$$

Using the fact that $(\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right)$ is a constant vector, we can rewrite the first term as

$$\begin{aligned} & \frac{1}{\gamma} cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}, \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}} \right)^T (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \right) \\ &= \frac{1}{\gamma} cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}, \left(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}} \right) \right)^T (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \\ &= \frac{1}{\gamma} \left(\Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right) (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right). \end{aligned}$$

For the second term, we can replace $(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1})$ with $(\mu_{\bar{Q}}^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}})$ because both $\mu_{\bar{Q}}^{\mathbf{r}}$ and $r_f \mathbf{1}$ are constant vectors, so this term is exactly the same as the first term. Therefore, we have

$$cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_{\bar{Q}}^{\mathbf{w}^d} \right) = \frac{2}{\gamma} \left(\Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right) (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right)$$

Under the assumption that $\Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} = v \Sigma_P^{\mathbf{r}}$,

$$cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_{\bar{Q}}^{\mathbf{w}^d} \right) = \frac{2}{\gamma} (v \Sigma_P^{\mathbf{r}}) (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) = \frac{2v}{\gamma} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right),$$

and the investor's portfolio is

$$\begin{aligned} \mathbf{w}^o &= \left(\gamma \Sigma_{\bar{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_{\bar{Q}}^{\mathbf{r}}} \right)^{-1} \left[\left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) - (\theta + \gamma) \left(\frac{\delta}{1 - \delta} \right) cov_{\pi} \left(\mu_{\bar{Q}}^{\mathbf{r}}, R_{\bar{Q}}^{\mathbf{w}^d} \right) \right] \\ &= (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \left[\left(\frac{1}{\gamma + v\theta} \right) - \left(\frac{\gamma + \theta}{\gamma + v\theta} \right) \left(\frac{\delta}{1 - \delta} \right) \frac{2v}{\gamma} \right] \end{aligned}$$

Using the simplified expression of $cov_\pi \left(\mu_Q^r, R_Q^{w^d} \right)$ and $\Sigma_\pi^{\mu_Q^r} = v\Sigma_P^r$, we have

$$\begin{aligned}
A &= cov_\pi \left(\mu_Q^r, R_Q^{w^d} \right)^T \left(\gamma\Sigma_Q^r + \theta\Sigma_\pi^{\mu_Q^r} \right)^{-1} cov_\pi \left(\mu_Q^r, R_Q^{w^d} \right) \\
&= \left(\frac{\gamma}{\gamma + v\theta} \right) \left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) \frac{4v^2}{\gamma^2} = \left(\frac{\gamma}{\gamma + v\theta} \right) R_Q^{w^d} \frac{4v^2}{\gamma^2} \\
B &= cov_\pi \left(\mu_Q^r, R_Q^{w^d} \right)^T \left(\gamma\Sigma_Q^r + \theta\Sigma_\pi^{\mu_Q^r} \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) = \left(\frac{\gamma}{\gamma + v\theta} \right) R_Q^{w^d} \frac{2v}{\gamma} \\
C &= \left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_Q^r + \theta\Sigma_\pi^{\mu_Q^r} \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) = \left(\frac{\gamma}{\gamma + v\theta} \right) R_Q^{w^d}
\end{aligned}$$

Next, we solve

$$\begin{aligned}
&E_\pi \left(R_Q^{w^d} \right) \\
&= E_\pi \left(\left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) \right) \\
&= E_\pi \left(\left(\mu_Q^r - \mu_Q^r + \mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r + \mu_Q^r - r_f \mathbf{1} \right) \right) \\
&= E_\pi \left(\left(\mu_Q^r - \mu_Q^r \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r \right) \right) + E_\pi \left(\left(\mu_Q^r - \mu_Q^r \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) \right) + \\
&E_\pi \left(\left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r \right) \right) + E_\pi \left(\left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right) \right),
\end{aligned}$$

where the second and third terms are zero because $E_\pi \left(\mu_Q^r - \mu_Q^r \right) = E_\pi \left(\mu_Q^r \right) - \mu_Q^r = 0$. The last term is the expected delegation return under the investor's average model, $R_Q^{w^d} = \left(\mu_Q^r - r_f \mathbf{1} \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - r_f \mathbf{1} \right)$. Therefore, we have

$$\begin{aligned}
E_\pi \left(R_Q^{w^d} \right) &= E_\pi \left(\left(\mu_Q^r - \mu_Q^r \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r \right) \right) + R_Q^{w^d} \\
&= tr \left[\left(\gamma\Sigma_P^r \right)^{-1} E_\pi \left(\left(\mu_Q^r - \mu_Q^r \right) \left(\mu_Q^r - \mu_Q^r \right)^T \right) \right] + R_Q^{w^d} \\
&= tr \left[\left(\gamma\Sigma_P^r \right)^{-1} \left(v\Sigma_P^r \right) \right] + R_Q^{w^d} \\
&= \frac{v}{\gamma} N + R_Q^{w^d}
\end{aligned}$$

Another way to solve $E_\pi \left(R_Q^{w^d} \right)$ is to notice that under the assumption $\Sigma_\pi^{\mu_Q^r} = v\Sigma_P^r$, $\left(\mu_Q^r - \mu_Q^r \right)^T \left(\gamma\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r \right) = \frac{v}{\gamma} \left(\mu_Q^r - \mu_Q^r \right)^T \left(v\Sigma_P^r \right)^{-1} \left(\mu_Q^r - \mu_Q^r \right)$ is a multiple of squared normalized Gaussian variable that has a Chi-squared distribution with the degree of freedom

equal to N and mean equal to $\frac{v}{\gamma}N$. Since $R_Q^{\mathbf{w}^d}$ can be decomposed,

$$\begin{aligned} R_Q^{\mathbf{w}^d} &= \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) + \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right) + \\ &\quad \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) + \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right), \\ &= \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) + 2 \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) + R_Q^{\mathbf{w}^d}, \end{aligned}$$

and the second term has zero mean, we have

$$R_Q^{\mathbf{w}^d} = \frac{v}{\gamma}N + R_Q^{\mathbf{w}^d}.$$

Similarly, to solve $\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d}\right)$, we also use the decomposition of $R_Q^{\mathbf{w}^d}$. The first term has a Chi-squared distribution, so its variance is

$$\sigma_\pi^2 \left(\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \right) = \frac{v^2}{\gamma^2} 2N.$$

The second term is a linear transformation of normal variable $\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)$ that has variance equal to $\Sigma_\pi^{\mu_Q^{\mathbf{r}}} = v \Sigma_P^{\mathbf{r}}$, so the second term's variance is

$$\sigma_\pi^2 \left(\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \right) = 2 \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} v \Sigma_P^{\mathbf{r}} (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right) 2 = 4 \frac{v}{\gamma} R_Q^{\mathbf{w}^d}.$$

The third term is a constant, the expected delegation return under the average model, so its' variance is zero. To solve $\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d}\right)$, we still need the covariance between the Chi-squared first component of $R_Q^{\mathbf{w}^d}$ and the Gaussian second component of $R_Q^{\mathbf{w}^d}$. First, we notice that the first moment of their product is zero:

$$\begin{aligned} &E_\pi \left(\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) 2 \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \right) \\ &= 2 \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} E_\pi \left(\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \right) \\ &= \frac{2v}{\gamma} \left(\mu_{\bar{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T (\gamma \Sigma_P^{\mathbf{r}})^{-1} E_\pi \left(\left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right)^T (v \Sigma_P^{\mathbf{r}})^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \left(\mu_Q^{\mathbf{r}} - \mu_{\bar{Q}}^{\mathbf{r}}\right) \right), \end{aligned}$$

which is equal to zero because by Isserlis' theorem, the expectation of three zero-mean multivariate normal variables is zero. Since the second component has zero mean, the product

of its and the first term's first moments is also zero. Therefore, the covariance between the first and second components of $R_Q^{\mathbf{w}^d}$ is zero. To sum up,

$$\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right) = \frac{v^2}{\gamma^2} 2N + 4 \frac{v}{\gamma} R_Q^{\mathbf{w}^d}.$$

Substitute the solutions of A , B , C , $E_\pi \left(R_Q^{\mathbf{w}^d} \right)$, and $\sigma_\pi^2 \left(R_Q^{\mathbf{w}^d} \right)$ into the optimal δ , we have

$$\begin{aligned} \delta &= \frac{E_\pi \left(R_Q^{\mathbf{w}^d(q)} \right) - (\theta + \gamma) B - C - \psi}{E_\pi \left(R_Q^{\mathbf{w}^d(q)} \right) + \theta \sigma_\pi^2 \left(R_Q^{\mathbf{w}^d(q)} \right) - (\theta + \gamma)^2 A - 2(\theta + \gamma) B - C} \\ &= \frac{\frac{v}{\gamma} N - \psi + \left[1 - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right) \right] R_Q^{\mathbf{w}^d}}{\frac{v}{\gamma} N + \theta \frac{v^2}{\gamma^2} 2N + \left[1 + 4 \frac{\theta v}{\gamma} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right)^2 \right] R_Q^{\mathbf{w}^d}} \end{aligned}$$

Comparative statics. The optimal delegation level is given by

$$\delta = \frac{\frac{v}{\gamma} N - \psi + \left[1 - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right) \right] R_Q^{\mathbf{w}^d}}{\left(1 + 2 \frac{\theta v}{\gamma} \right) \frac{v}{\gamma} N + \left[1 + 4 \frac{\theta v}{\gamma} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right)^2 \right] R_Q^{\mathbf{w}^d}}$$

Under the three special conditions, we prove the following results of comparative statics. First, we prove that $\frac{\partial \delta}{\partial N} > 0$. Note that

$$1 - \delta = \frac{\frac{2\theta v}{\gamma} \frac{v}{\gamma} N + \psi + \left[4 \frac{\theta v}{\gamma} - \frac{2v(\theta + \gamma)}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right) \right] R_Q^{\mathbf{w}^d}}{\left(1 + 2 \frac{\theta v}{\gamma} \right) \frac{v}{\gamma} N + \left[1 + 4 \frac{\theta v}{\gamma} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right)^2 \right] R_Q^{\mathbf{w}^d}},$$

so as N increases, $1 - \delta$ declines because the coefficient of N is larger in the denominator, and therefore, δ increases. Next, multiplying the numerator and denominator of δ by $\frac{\gamma}{vN}$, we have

$$\delta = \frac{1 + \frac{\gamma}{vN} \left(R_Q^{\mathbf{w}^d} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right) R_Q^{\mathbf{w}^d} - \psi \right)}{\left(1 + 2 \frac{\theta v}{\gamma} \right) + \frac{\gamma R_Q^{\mathbf{w}^d}}{vN} \left(1 + 4 \frac{\theta v}{\gamma} - \frac{\gamma}{\gamma + v\theta} \left(\frac{2v(\theta + \gamma)}{\gamma} + 1 \right)^2 \right)},$$

so when N is sufficiently large, what determines δ is $\left(1 + 2 \frac{\theta v}{\gamma} \right)$ in the denominator. Therefore, we have $\frac{\partial \delta}{\partial v} < 0$, $\frac{\partial \delta}{\partial \theta} < 0$ and $\frac{\partial \delta}{\partial \gamma} > 0$.

Now we prove that given δ , $\frac{\partial \mathbf{w}^o}{\partial \theta} < 0$, $\frac{\partial \mathbf{w}^o}{\partial v} < 0$, and $\frac{\partial \mathbf{w}^o}{\partial \gamma} < 0$. The investor's portfolio is

$$\mathbf{w}^o = (\Sigma_P^{\mathbf{r}})^{-1} \left(\mu_{\mathcal{Q}}^{\mathbf{r}} - r_f \mathbf{1} \right) \left[\frac{1}{\gamma + v\theta} - 2v \left(\frac{\gamma + \theta}{\gamma + v\theta} \right) \frac{1}{\gamma} \left(\frac{\delta}{1 - \delta} \right) \right].$$

First, note that since $v < 1$, $\frac{\gamma + \theta}{\gamma + v\theta}$ increases in θ , so $\frac{\partial \mathbf{w}^o}{\partial \theta} < 0$. Moreover, since $\gamma + \delta > \theta$, $v \left(\frac{\gamma + \theta}{\gamma + v\theta} \right)$ increases in v , so $\frac{\partial \mathbf{w}^o}{\partial v} < 0$. Finally, note that $\left(\frac{\gamma + \theta}{\gamma + v\theta} \right) \frac{1}{\gamma} = \frac{1 + \theta/\gamma}{1 + v\theta/\gamma}$, decreasing in γ , while $\frac{1}{\gamma + v\theta}$ also decreases in γ . Therefore, the effect of γ is ambiguous.

Appendix IV: Delegating under Model Uncertainty

The model solves the optimal delegation level δ . It can serve as a normative framework to guide the choice of delegation for investors and institutions (e.g., pension funds). The necessary inputs are the preference parameters, management fee, and the structure of model uncertainty, which we show can be extracted from the Bayesian posterior of models of asset returns. Next, we show the model-implied δ together with the delegation level in data.

We plot the optimal delegation δ under simulated ambiguity (more details later), and the detrended empirical counterpart in Figure A.2. We detrend the fund ownership data because the rise of fund ownership may be due to technological progress or the evolution of stock market composition that are outside of our model. While the scales are different, the model-implied and empirical δ are reasonably correlated. The correlations are 0.19 and 0.14 respectively with linearly detrended and HP-filtered empirical series.

Next, we lay out the details on how to calculate model-implied δ . What we require is a set of candidate distributions and investors' prior over these distributions (π). For this reason, we specify a concrete set of return distributions generated from a latent-state model, and use the Bayesian posterior as π . The output depends on this specific approach to generate Δ and π . This exercise only serves as an illustration of how the model produces the optimal level of delegation by incorporating model uncertainty that an investor faces.

[Insert Figure A.2 here.]

Parameters and assets. The risk aversion γ is set to 2, and ambiguity aversion θ is set to 8.864. Both are chosen by Ju and Miao (2012) to match the risk-free rate and the equity

premium under smooth ambiguity averse preference. The management fee ψ is 1%, in line with the asset management cost in the U.S. equity market (French (2008)). The risk-free rate is the one-month Treasury-bill rate. Returns of risky assets are monthly returns of the six size and book-to-market sorted portfolios from Kenneth French’s website.

Ambiguity Structure. The investor holds the belief that asset returns are drawn from a normal distribution $N(\theta, \Sigma_t^r)$ with constant mean θ and time-varying covariance matrix Σ^{r_t} .²⁵ The covariance matrix is decomposed into a time-invariant idiosyncratic part Ω , and a time-varying part BH_tB^T , where B is a constant matrix and H_t is a K -dimension diagonal matrix $diag\left(\{h_{k,t}\}_{k=1}^K\right)$ whose elements follow log-AR(1) process with i.i.d. normal shocks:

$$\log(h_{k,t}) = \alpha_k + \delta_k \log(h_{k,t-1}) + \sigma_k^v v_{k,t}, \quad v_{k,t} \sim i.i.d.N(0, 1) \quad (41)$$

This is the dynamic factor model of multivariate stochastic volatility studied by Jacquier, Polson, and Rossi (1999) and Aguilar and West (2000).

Therefore, each return model, $N(\theta, \Sigma^{r_t}) \in \Delta$, is indexed by the values of parameters $(\alpha_k, \delta_k, \sigma_k^v)$ and latent states $(h_{k,t})$. The uncertainty in these quantities spans the representative investor’s model space Δ . There are two sources of ambiguity in return distribution: (1) parameter uncertainty; (2) latent state uncertainty. The first source declines over time as data accumulate, while the second does not. The investor learns the parameters and updates her belief over values of state variables over time, having in mind this structure of ambiguity. We calculate the posterior probability distribution of $N(\theta, \Sigma^{r_t})$, and input the posterior statistics in the closed-form solution of optimal delegation given by Equation (37).

In the implementation, we assume $K = 1$. Investors’ belief π_t is updated from August 1983 to September 2012 (350 months). The previous 685 months (July 1926 to July 1983) is used as a training set to form the initial prior π_1 based on the smoothing algorithm (Gibbs sampler). The learning problem is solved by “particle filter”, a recursive algorithm commonly used to estimate non-linear latent factor models. Due to its complexity, we provide the details on the estimation and calculation in a separate technical report available upon request.

²⁵Among many studies, Bossaerts and Hillion (1999) compare a variety of stock return predictors and conclude that even the best prediction models have no out-of-sample forecasting power. Pesaran and Timmermann (1995) argue that predictability of stock returns is very low. Henriksson (1984) and Ferson and Schadt (1996) among others show that most mutual funds are not successful return timers. Following these studies, we assume constant expected return θ .

Discussion: managers' knowledge. In the theoretical model, fund managers know the true probability distribution of returns. So, in the current setting, investors believe that the fund managers know exactly the true $N(\theta, \Sigma^r)$. In other words, at time t the fund manager's knowledge includes not only the parameters $(\theta, B, \Omega$ and $\{(\alpha_k, \delta_k, \sigma_k^v)\}_{k=1}^K$) but also the true value of latent states H_t .²⁶ The predictability of stock volatility has been shown by Andersen, Bollerslev, Christoffersen, and Diebold (2006) among others. Studies, such as Johannes, Korteweg, and Polson (2014) and Marquering and Verbeek (2004), demonstrate that volatility timing can add value to investors' portfolios. Busse (1999) shows that mutual fund managers time conditional market return volatility, and Chen and Liang (2007) show the same for hedge funds. Fund managers' ability to know the true parameter values and observe the volatilities is the extreme version of volatility timing. Investors' learning of H_t already exhibits a certain level of volatility timing, but investors assume that fund managers can do even better thanks to better econometric models and access to more data sources.

²⁶These are two extreme cases of knowledge. In the middle of the spectrum, for example, we may assume that investors understand the model structure but do not know the parameter values and state values, while fund managers know the model structure and parameter values but do not observe directly the state variable. The key is that the fund managers face less model uncertainty than the investors do.

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Table 1 Summary Statistics

	ACR	HML	BAB	CMA	IVOL	LTR	MOM	NI	RMW	STR	TVOL
Panel A: Factor returns (annualized)											
count	447	447	447	447	447	447	447	447	447	447	447
mean	0.01	0.03	0.01	0.03	0.04	0.03	0.07	0.04	0.03	0.05	0.04
std	0.18	0.36	0.59	0.24	0.53	0.30	0.54	0.31	0.32	0.40	0.59
25%	-0.10	-0.19	-0.29	-0.12	-0.23	-0.16	-0.13	-0.11	-0.09	-0.14	-0.27
50%	0.01	0.01	0.01	0.02	0.02	0.01	0.07	0.02	0.03	0.02	0.03
75%	0.11	0.22	0.37	0.17	0.30	0.19	0.34	0.17	0.17	0.23	0.33
ρ	0.21	0.15	0.04	0.13	0.12	0.18	0.07	0.14	0.10	-0.03	0.08
Panel B: Factor institutional ownership <i>INST</i> (%)											
mean	-0.31	-0.75	-2.61	-1.01	-1.43	-1.00	0.69	-0.78	-0.92	-0.19	-1.69
std	0.70	1.00	1.50	0.89	1.53	1.70	1.45	1.30	0.86	1.49	1.37
25%	-0.73	-1.30	-3.26	-1.47	-2.37	-2.08	-0.09	-1.69	-1.46	-0.99	-2.60
50%	-0.36	-0.66	-2.28	-0.84	-1.50	-0.58	0.66	-0.71	-0.86	-0.16	-1.68
75%	-0.08	0.05	-1.54	-0.38	-0.75	0.09	1.55	-0.13	-0.43	0.69	-0.97
ρ	0.86	0.86	0.94	0.86	0.73	0.94	0.83	0.90	0.90	-0.04	0.64
Panel C: Factor individual ownership <i>INDV</i> (%)											
mean	0.90	3.23	9.70	1.02	0.38	1.75	-2.08	-0.62	-1.23	1.29	1.35
std	1.90	2.98	3.90	2.15	3.95	4.32	4.25	3.66	2.30	4.52	3.69
25%	-0.23	0.91	7.25	-0.57	-2.27	-1.19	-5.30	-3.48	-2.62	-1.84	-1.09
50%	0.90	3.18	10.04	1.51	0.17	1.34	-1.91	-0.85	-1.11	1.40	1.13
75%	2.04	5.50	12.49	2.67	2.84	4.39	1.02	2.39	0.41	4.68	3.46
ρ	0.91	0.95	0.95	0.91	0.75	0.94	0.82	0.94	0.94	-0.02	0.74
Panel D: Uncertainty measures and survey expectation											
	<i>U^{CSA}</i>	<i>U^{PCA}</i>	<i>EPU</i>	<i>EPU^{news}</i>	<i>VIX</i>	<i>VXO</i>	<i>AA</i>				
count	384	384	387	387	327	375	357				
mean	0.95	0.95	107.96	109.80	19.64	20.45	0.08				
std	0.06	0.06	32.00	40.35	7.48	8.25	0.15				
25%	0.92	0.92	84.42	82.32	13.85	14.30	-0.02				
50%	0.94	0.94	100.74	99.78	17.83	18.60	0.09				
75%	0.96	0.97	125.15	126.29	23.59	24.30	0.18				
ρ	0.99	0.99	0.82	0.68	0.84	0.83	0.61				

Note. This table shows the number of observations, mean, standard deviation, quintile values, and autocorrelation coefficient (ρ) of monthly returns (Panel A), institutional ownership (Panel B), and individual ownership (Panel C) for each factor. The construction of the long-short factors returns and institutional ownership follows the [Fama and French \(1993\)](#) procedure and is described in details in the main text. Panel D reports the summary statistics of the six uncertainty measures (details in the main text) and the survey data on investors' expectations.

Table 2 Factor Alphas by the Correlation between Individual Ownership and Uncertainty

	Full-sample α			Full-sample $\alpha \delta$		
	H	L	L-H	H	L	L-H
$\hat{\alpha}$	0.08%	2.67%	2.59%	2.26%	5.18%	2.92%
t -value	[0.13]	[2.03]	[2.57]	[2.28]	[2.75]	[1.84]

Note. This table reports statistics of CAPM alphas of factors ranked by the full-sample correlations between factor individual ownership, $INDV$, and uncertainty measure, \mathcal{U}^{PCA} , from [Jurado, Ludvigson, and Ng \(2015\)](#). Portfolio “H” contains the equal-weighted 6 high correlations factors and portfolio “L” contains equal-weighted 5 low correlations factors. Portfolio “L-H” longs L portfolio and shorts H portfolio. CAPM alphas are the intercept of time-series regression of portfolio excess returns on the market excess returns. Excess returns are defined as the difference between raw returns and the monthly risk-free rate. The left panel reports full-sample CAPM α and the right panel reports α controlling for aggregate institutional ownership, i.e., δ in the model. α estimates are annualized and t -statistics are reported in the brackets.

Table 3 Uncertainty and the Future Dispersion of Factor CAPM Residuals

	$Dispersion_t = \epsilon_{i,t}^{max} - \epsilon_{i,t}^{min}$						$Dispersion_t = \sigma_t(\epsilon_{i,t})$					
	\mathcal{U}^{CSA}	\mathcal{U}^{PCA}	EPU	EPU^{news}	VIX	VXO	\mathcal{U}^{CSA}	\mathcal{U}^{PCA}	EPU	EPU^{news}	VIX	VXO
Panel A: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \epsilon_t$												
γ_1	1.66*** (6.43)	1.34*** (5.10)	0.80*** (3.06)	1.15*** (4.48)	2.54*** (9.51)	2.29*** (9.41)	0.46*** (5.90)	0.36*** (4.58)	0.20** (2.51)	0.31*** (3.92)	0.73*** (8.88)	0.66*** (8.86)
N	385	385	386	386	326	374	385	385	386	386	326	374
R^2	0.10	0.06	0.02	0.05	0.22	0.19	0.08	0.05	0.02	0.04	0.20	0.17
Panel B: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \gamma_2 \delta_{t-1} + \epsilon_t$												
γ_1	1.65*** (6.39)	1.67*** (6.21)	0.76*** (2.88)	1.16*** (4.34)	2.55*** (9.44)	2.34*** (9.56)	0.45*** (5.85)	0.46*** (5.74)	0.19** (2.32)	0.31*** (3.78)	0.73*** (8.80)	0.67*** (8.98)
γ_2	0.12*** (2.64)	0.21*** (4.41)	0.05 (1.05)	0.02 (0.57)	0.01 (0.23)	0.09** (2.19)	0.04*** (2.94)	0.07*** (4.55)	0.02 (1.15)	0.01 (0.69)	0.00 (0.27)	0.03** (2.14)
N	384	384	384	384	324	372	384	384	384	384	324	372
R^2	0.11	0.11	0.03	0.05	0.22	0.20	0.10	0.10	0.02	0.04	0.19	0.18
Panel C: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \gamma_2 \delta_{t-1}^{detrend} + \epsilon_t$												
γ_1	2.08*** (6.55)	1.49*** (4.79)	0.80*** (3.04)	1.20*** (4.61)	2.56*** (9.50)	2.31*** (9.46)	0.58*** (6.05)	0.40*** (4.31)	0.20** (2.49)	0.32*** (4.06)	0.73*** (8.86)	0.67*** (8.89)
γ_2	-0.46** (-2.28)	-0.18 (-0.91)	0.10 (0.56)	0.17 (0.94)	0.18 (1.03)	0.25 (1.48)	-0.13** (-2.19)	-0.05 (-0.87)	0.03 (0.62)	0.05 (0.93)	0.06 (1.04)	0.07 (1.45)
N	384	384	384	384	324	372	384	384	384	384	324	372
R^2	0.11	0.06	0.03	0.05	0.22	0.20	0.09	0.05	0.02	0.04	0.20	0.18

Note. This table reports the results of forecasting the cross-section dispersions of factor CAPM residuals using uncertainty measures in Panel A. Panel B controls for the raw institutional ownership, δ ; Panel C controls for the detrended institutional ownership, $\delta^{detrend}$. We measure dispersion in two ways: (1) the cross-section difference between maximum and minimum (on the left); (2) the cross-sectional standard deviation (on the right). The uncertainty measures include the cross-sectional average (\mathcal{U}^{CSA}) and first principal component (\mathcal{U}^{PCA}) of uncertainties estimated using a large set of macro and financial variables from [Jurado, Ludvigson, and Ng \(2015\)](#); baseline Economic Policy Uncertainty (EPU) and news-based Economic Policy Uncertainty (EPU^{news}) from [Baker, Bloom, and Davis \(2016\)](#); CBOE stock market volatility indexes VIX and VXO . For each specification, the sample size is determined by the availability of the uncertainty measure and fund ownership. All uncertainty measures are normalized to have mean of 0 and standard deviation of 1%. *, **, and *** indicate 10%, 5% and 1% statistical significance respectively.

Table 4 Predicting Future Factor Returns with Fund Ownership

Panel A: panel regressions						
	$R_{i,t+1}^{3m}$					
	(1)	(2)	(3)	(4)	(5)	(6)
<i>INST</i>	0.27*** (0.10)	0.21** (0.11)	0.27*** (0.10)	0.31*** (0.09)	0.22* (0.10)	0.28** (0.11)
<i>RV</i>				0.24 (0.22)	0.02 (0.19)	0.28 (0.30)
Constant	0.01*** (0.00)			0.00 (0.01)		
Quarter FE		✓			✓	
Factor FE			✓			✓
Observations	4,884	4,884	4,884	4,513	4,513	4,513
Adjusted R^2	0.00	0.22	0.00	0.01	0.22	0.01
Residual Std. Error	0.06	0.06	0.06	0.06	0.06	0.06

Panel B: Hodrick (1992) reverse predictive regressions						
	$3 \times R_{i,t+1}^{1m}$					
	(1)	(2)	(3)	(4)	(5)	(6)
$\frac{1}{3} \sum_{j=0}^2 INST_{t-j}^n$	0.31*** (0.11)	0.28** (0.11)	0.32** (0.13)	0.36*** (0.11)	0.29*** (0.11)	0.34*** (0.13)
<i>RV</i>				0.26*** (0.08)	0.03 (0.11)	0.30*** (0.10)
Constant	0.01*** (0.00)			0.00 (0.00)		
Quarter FE		✓			✓	
Factor FE			✓			✓
Observations	4,884	4,884	4,884	4,535	4,535	4,535
Adjusted R^2	0.00	0.06	0.00	0.00	0.06	0.00
Residual Std. Error	0.10	0.10	0.10	0.11	0.10	0.11

Note. This table shows predictive regressions of monthly long–short factor returns on lagged values of the factor (relative) institutional ownership (*INST*) controlling for other factor return predictors such as realized volatility *RV*. Panel A reports estimations from pooled OLS and fixed effect panel regressions:

$$R_{i,t+1}^{3m} = \alpha + \beta \cdot INST_{i,t} + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1}$$

The left hand variable is monthly overlapping 3-month returns. Since ownership data is refreshed quarterly, standard errors are double-clustered at quarter and factor levels. Panel B reports estimations using Hodrick reverse predictive regressions

$$3 \times R_{i,t+1}^{1m} = \alpha + \beta \left(\frac{1}{3} \sum_{j=0}^2 INST_{i,t-j}^n \right) + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1}^n$$

The left hand variable is monthly non-overlapping returns multiplied by a factor of 3 to be compared with estimates from Panel A. The sample period is 198003:201612. Standard errors are in parentheses. *, **, and *** indicate 10%, 5% and 1% statistical significance respectively.

Table 5 Summary Statistics: Equal-weighted Portfolios of Factors by Fund Ownership

	H	M	L	H-L
Mean (ann.)	4.98%	2.68%	2.06%	2.91%
Vol (ann.)	6.16%	7.34%	11.38%	10.93%
Sharpe	0.81	0.36	0.18	0.27
Skewness	-0.18	-0.71	-0.42	0.96
Kurtosis	13.03	10.12	4.85	7.56
Observations	444	444	444	444

Note. This table reports the annualized mean, volatility, Sharpe ratio, skewness, and kurtosis of the returns of factor portfolios. We sort factors by their institutional ownership, *INST*, at the end of each quarter, and form equal-weighted high (top four factors), medium (3 factors) and low (bottom four factors) portfolios.

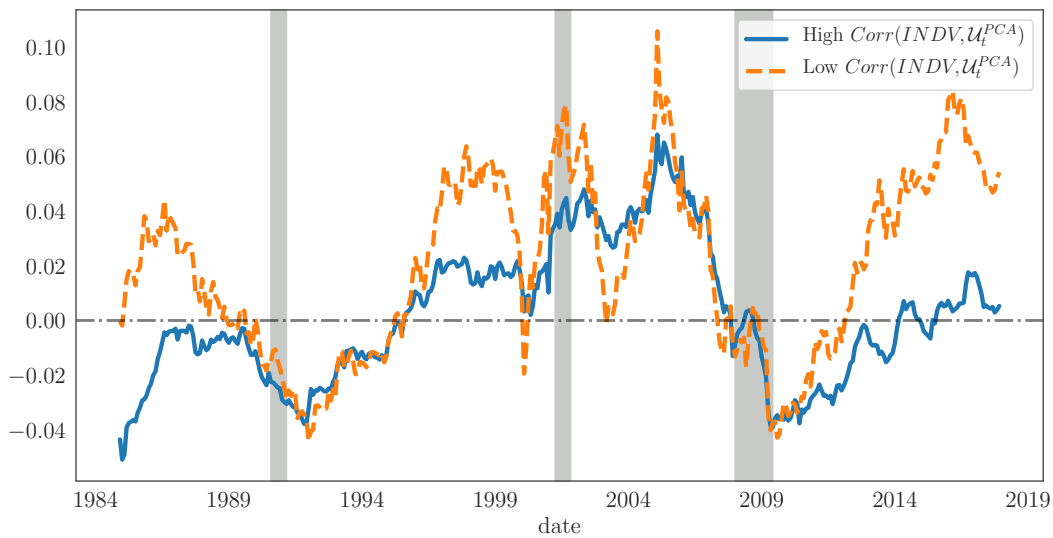


Figure 1 Rolling Factor Alphas by the Correlation between Individual Ownership and Uncertainty

This figure plots 60-month rolling CAPM alphas of factors sorted by correlations between individual ownership $INDV$ and uncertainty measure U^{PCA} from Jurado, Ludvigson, and Ng (2015). CAPM alphas are the intercept of time-series regression of portfolio excess returns on the market excess returns. Excess returns are defined as the difference between raw returns and the monthly risk-free rate. H portfolio contains the equal-weighted 6 high correlations factors. L portfolio contains equal-weighted 5 low correlations factors.

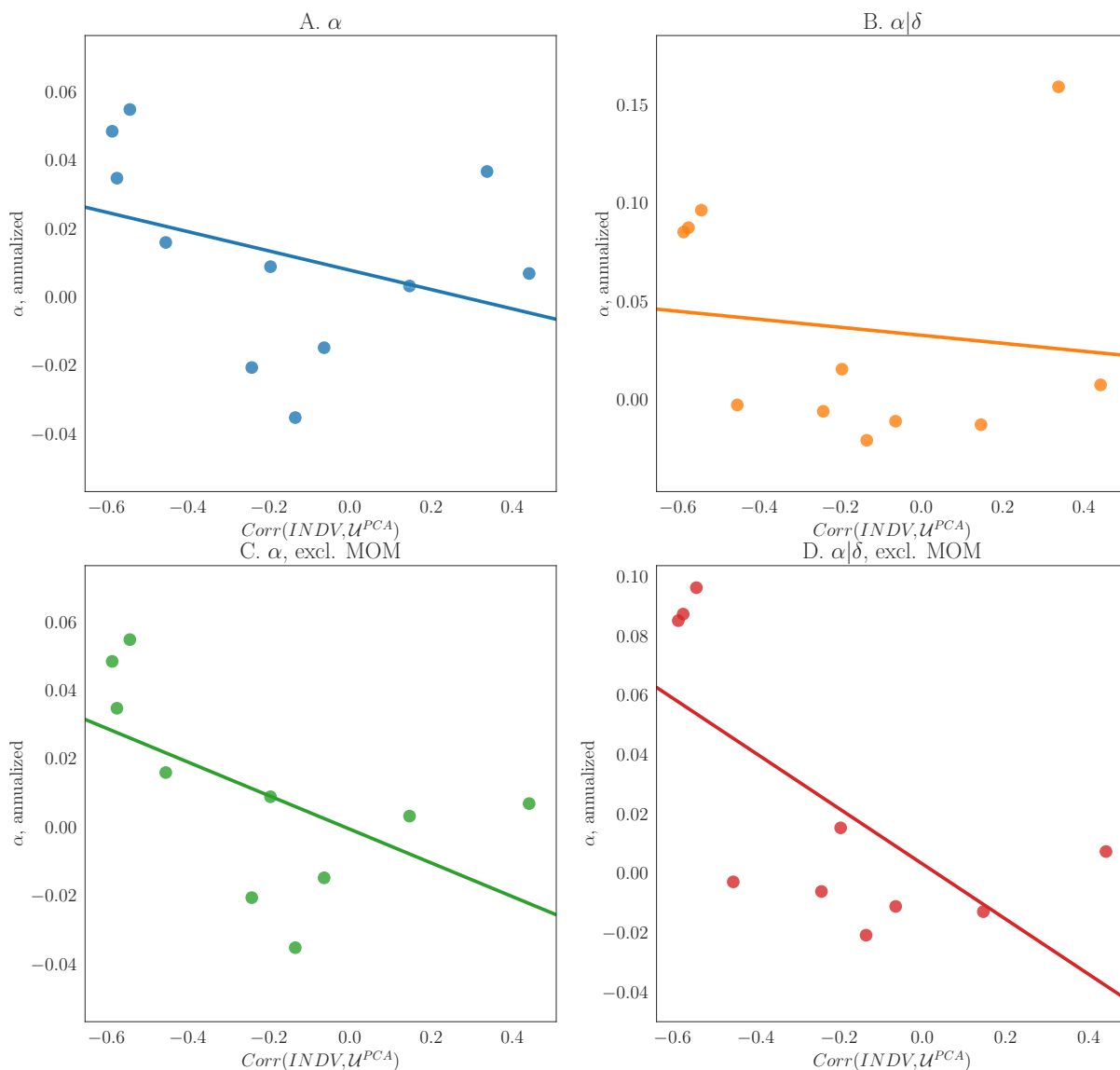


Figure 2 Factor Alphas and Correlations between Individual Ownership and Uncertainty

This figure plots the full-sample annualized alpha of each factor against the correlation between the factor's individual ownership, $INDV$, and uncertainty measure, U^{PCA} from Jurado, Ludvigson, and Ng (2015). Each dot represents one factor. Panel A and C plot raw CAPM alphas. Panel B and D plot CAPM alphas controlling for aggregate institutional ownership δ . Panel C and D exclude the momentum factor, MOM . CAPM alphas are the intercept of time-series regression of factor excess returns on the market excess returns. Excess returns are defined as the difference between raw returns and the monthly risk-free rate.

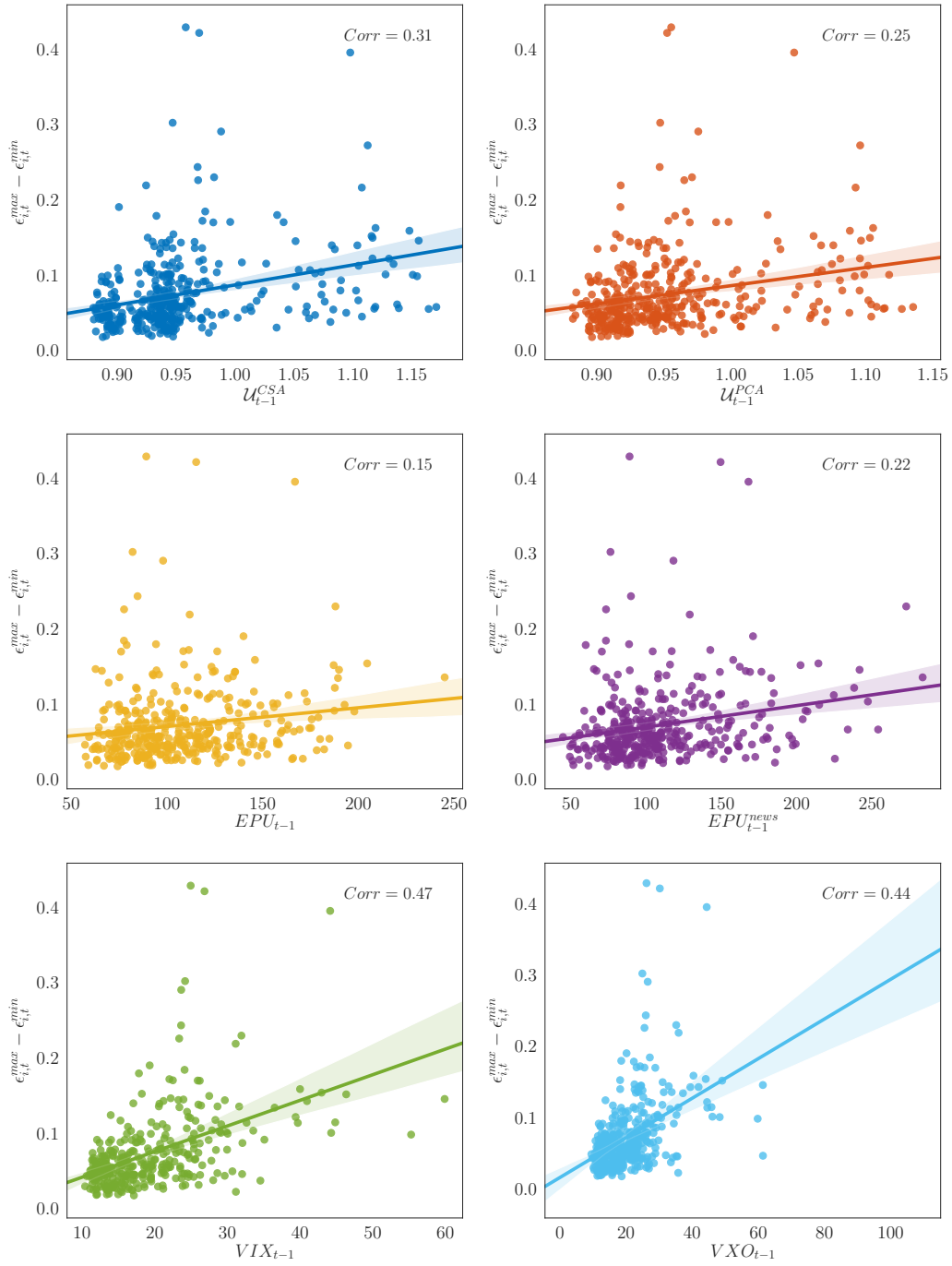


Figure 3 Uncertainty and the Future Dispersion of Factor CAPM Residuals

The dispersion is measured as the time t cross-sectional difference between maximum and minimum of factor's CAPM residuals. The correlation between an uncertainty measure and dispersion is shown top-right in each panel. The uncertainty measures include the cross-sectional average (U_t^{CSA}) and first principal component (U_t^{PCA}) of uncertainties estimated using a large set of macro and financial variables from [Jurado, Ludvigson, and Ng \(2015\)](#); baseline Economic Policy Uncertainty (EPU) and news-based Economic Policy Uncertainty (EPU^{news}) from [Baker, Bloom, and Davis \(2016\)](#); CBOE stock market volatility indexes VIX and VXO .

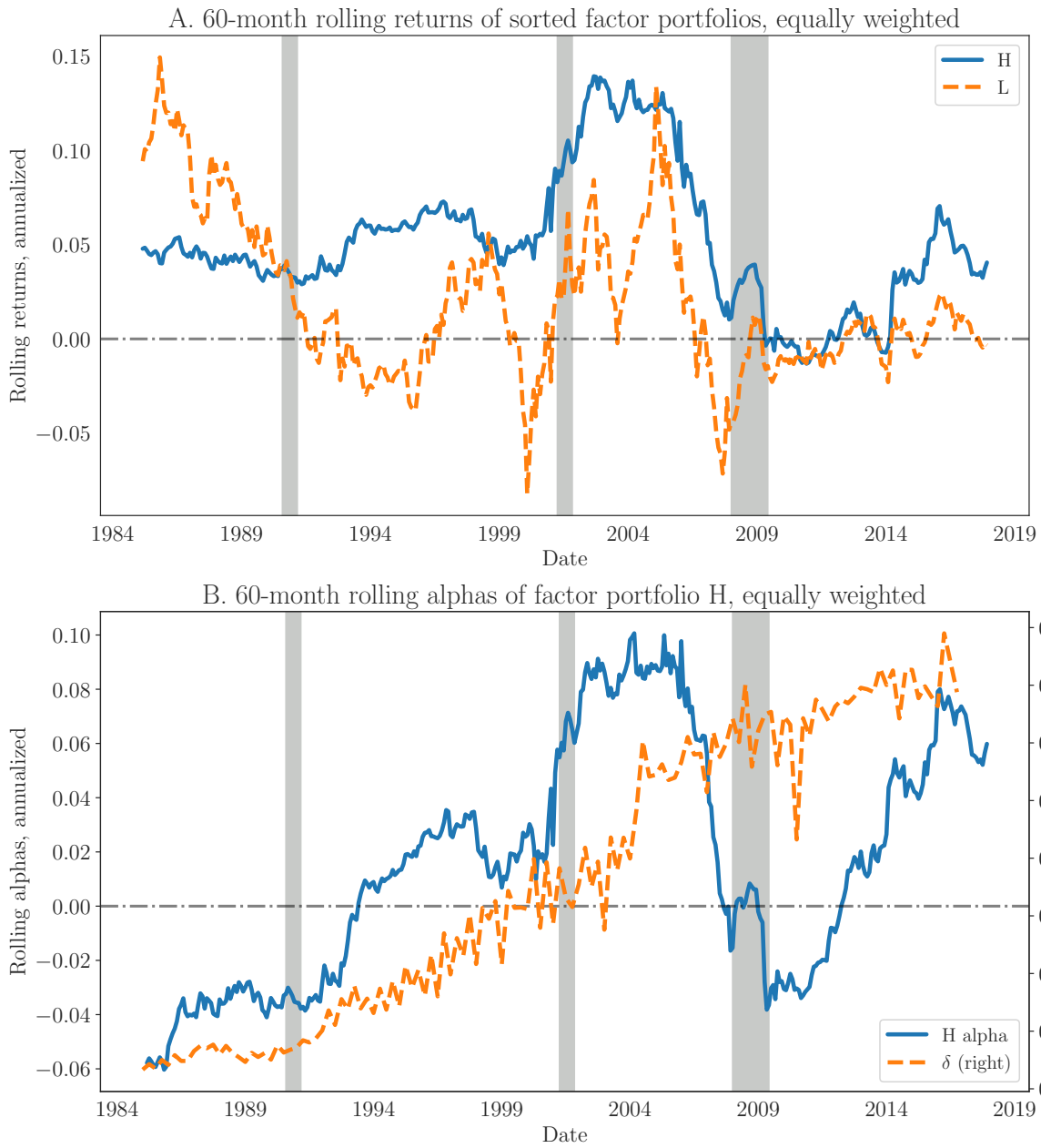


Figure 4 Rolling Average Returns and Alphas of Factors Sorted by Fund Ownership
 This figure plots the annualized rolling average returns and alphas of factors sorted by fund ownership, *INST*. Equal-weighted high (top four factors) and low (bottom four factors) portfolios are formed. Panel A plots the 60-month moving average of H and L portfolios' returns. Panel B plots the 60-month rolling-window estimates of H portfolio's CAPM alpha and the aggregate fund ownership, δ .

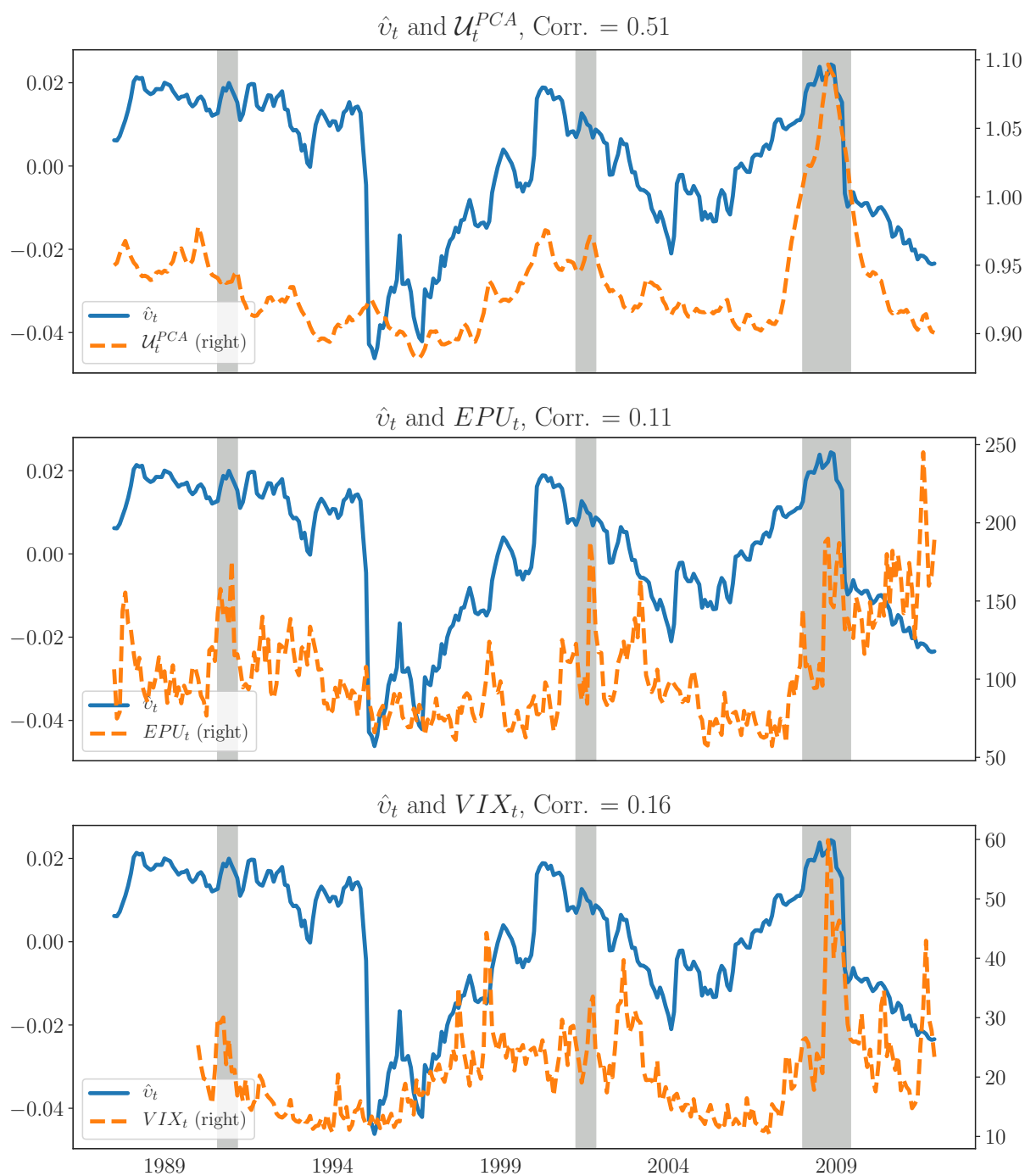


Figure 5 Model-implied and Alternative Uncertainty Measures

This figure plots the model-implied uncertainty (left Y-axis) estimated using a two-step procedure (details in the main text). Also shown are the other uncertainty measures (right Y-axis), including the first principal component (U_t^{PCA}) of uncertainties estimated from a large set of macro and financial variables by [Jurado, Ludvigson, and Ng \(2015\)](#), the Economic Policy Uncertainty (EPU) of [Baker, Bloom, and Davis \(2016\)](#), and CBOE volatility index (VIX). The correlations between the model-implied and other uncertainty measures are reported in the panel titles. The shaded areas mark the NBER recession periods.

Table A.1 Factor Alphas by the Correlation between Individual Ownership and Uncertainty (Alternative Measures)

	Full-sample α			Full-sample $\alpha \delta$		
	H	L	L-H	H	L	L-H
Panel A: Uncertainty = U^{CSA}						
$\hat{\alpha}$	-0.40%	3.26%	3.66%	1.91%	5.61%	3.70%
t -value	[-0.58]	[2.15]	[2.52]	[1.86]	[2.58]	[1.68]
Panel B: Uncertainty = EPU						
$\hat{\alpha}$	0.33%	2.37%	2.04%	2.51%	4.89%	2.38%
t -value	[0.44]	[1.83]	[1.88]	[2.27]	[2.65]	[1.42]
Panel C: Uncertainty = EPU^{news}						
$\hat{\alpha}$	-0.01%	2.79%	2.80%	2.35%	5.08%	2.73%
t -value	[-0.02]	[2.06]	[2.40]	[2.28]	[2.53]	[1.44]
Panel D: Uncertainty = VIX						
$\hat{\alpha}$	0.24%	2.48%	2.24%	2.09%	5.39%	3.29%
t -value	[0.35]	[1.88]	[2.19]	[2.19]	[2.77]	[1.97]
Panel E: Uncertainty = VXO						
$\hat{\alpha}$	0.24%	2.48%	2.24%	2.09%	5.39%	3.29%
t -value	[0.35]	[1.88]	[2.19]	[2.19]	[2.77]	[1.97]

Note. This table reports statistics of CAPM alphas of factors ranked by the full-sample correlations between factor individual ownership, $INDV$, and five alternative uncertainty measures introduced in Section 3.1. Portfolio “H” contains the equal-weighted 6 high correlations factors and portfolio “L” contains equal-weighted 5 low correlations factors. Portfolio “L-H” longs L portfolio and shorts H portfolio. CAPM alphas are the intercept of time-series regression of portfolio excess returns on the market excess returns. Excess returns are defined as the difference between raw returns and the monthly risk-free rate. The left panel reports full-sample CAPM α and the right panel reports α controlling for aggregate institutional ownership, i.e., δ in the model. α estimates are annualized and t -statistics are reported in the brackets.

Table A.2 Uncertainty and the Future Dispersion of Factor Returns

	$Dispersion_t = r_{i,t}^{max} - r_{i,t}^{min}$						$Dispersion_t = \sigma_t(r_{i,t})$					
	\mathcal{U}^{CSA}	\mathcal{U}^{PCA}	EPU	EPU^{news}	VIX	VXO	\mathcal{U}^{CSA}	\mathcal{U}^{PCA}	EPU	EPU^{news}	VIX	VXO
Panel A: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \epsilon_t$												
γ_1	1.76*** (6.12)	1.34*** (4.55)	1.10*** (3.80)	1.56*** (5.48)	3.21*** (11.11)	2.94*** (11.17)	0.55*** (6.10)	0.40*** (4.35)	0.32*** (3.49)	0.47*** (5.19)	1.01*** (11.09)	0.92*** (11.17)
N	385	385	386	386	326	374	385	385	386	386	326	374
R^2	0.09	0.05	0.04	0.07	0.28	0.25	0.09	0.05	0.03	0.07	0.27	0.25
Panel B: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \gamma_2 \delta_{t-1} + \epsilon_t$												
γ_1	1.65*** (6.39)	1.67*** (6.21)	0.76*** (2.88)	1.16*** (4.34)	2.55*** (9.44)	2.34*** (9.56)	0.45*** (5.85)	0.46*** (5.74)	0.19** (2.32)	0.31*** (3.78)	0.73*** (8.80)	0.67*** (8.98)
γ_2	0.12*** (2.64)	0.21*** (4.41)	0.05 (1.05)	0.02 (0.57)	0.01 (0.23)	0.09** (2.19)	0.04*** (2.94)	0.07*** (4.55)	0.02 (1.15)	0.01 (0.69)	0.00 (0.27)	0.03** (2.14)
N	384	384	384	384	324	372	384	384	384	384	324	372
R^2	0.11	0.11	0.03	0.05	0.22	0.20	0.10	0.10	0.02	0.04	0.19	0.18
Panel C: $Dispersion_t = \gamma_0 + \gamma_1 Uncertainty_{t-1} + \gamma_2 \delta_{t-1}^{detrend} + \epsilon_t$												
γ_1	2.08*** (6.55)	1.49*** (4.79)	0.80*** (3.04)	1.20*** (4.61)	2.56*** (9.50)	2.31*** (9.46)	0.58*** (6.05)	0.40*** (4.31)	0.20** (2.49)	0.32*** (4.06)	0.73*** (8.86)	0.67*** (8.89)
γ_2	-0.46** (-2.28)	-0.18 (-0.91)	0.10 (0.56)	0.17 (0.94)	0.18 (1.03)	0.25 (1.48)	-0.13** (-2.19)	-0.05 (-0.87)	0.03 (0.62)	0.05 (0.93)	0.06 (1.04)	0.07 (1.45)
N	384	384	384	384	324	372	384	384	384	384	324	372
R^2	0.11	0.06	0.03	0.05	0.22	0.20	0.09	0.05	0.02	0.04	0.20	0.18

Note. This table reports the results of forecasting the cross-section dispersions of factor returns using uncertainty measures in Panel A. Panel B controls for the raw institutional ownership, δ ; Panel C controls for the detrended institutional ownership, $\delta^{detrend}$. We measure dispersion in two ways: (1) the cross-section difference between maximum and minimum (on the left); (2) the cross-section standard deviation. The uncertainty measures include the cross-sectional average (\mathcal{U}^{CSA}) and first principal component (\mathcal{U}^{PCA}) of uncertainties estimated using a large set of macro and financial variables from Jurado, Ludvigson, and Ng (2015); baseline Economic Policy Uncertainty (EPU) and news-based Economic Policy Uncertainty (EPU^{news}) from Baker, Bloom, and Davis (2016); CBOE stock market volatility indexes VIX and VXO . For each specification, the sample size is determined by the availability of the uncertainty measure and fund ownership. All uncertainty measures are normalized to have mean of 0 and standard deviation of 1%. *, **, and *** indicate 10%, 5% and 1% statistical significance respectively.

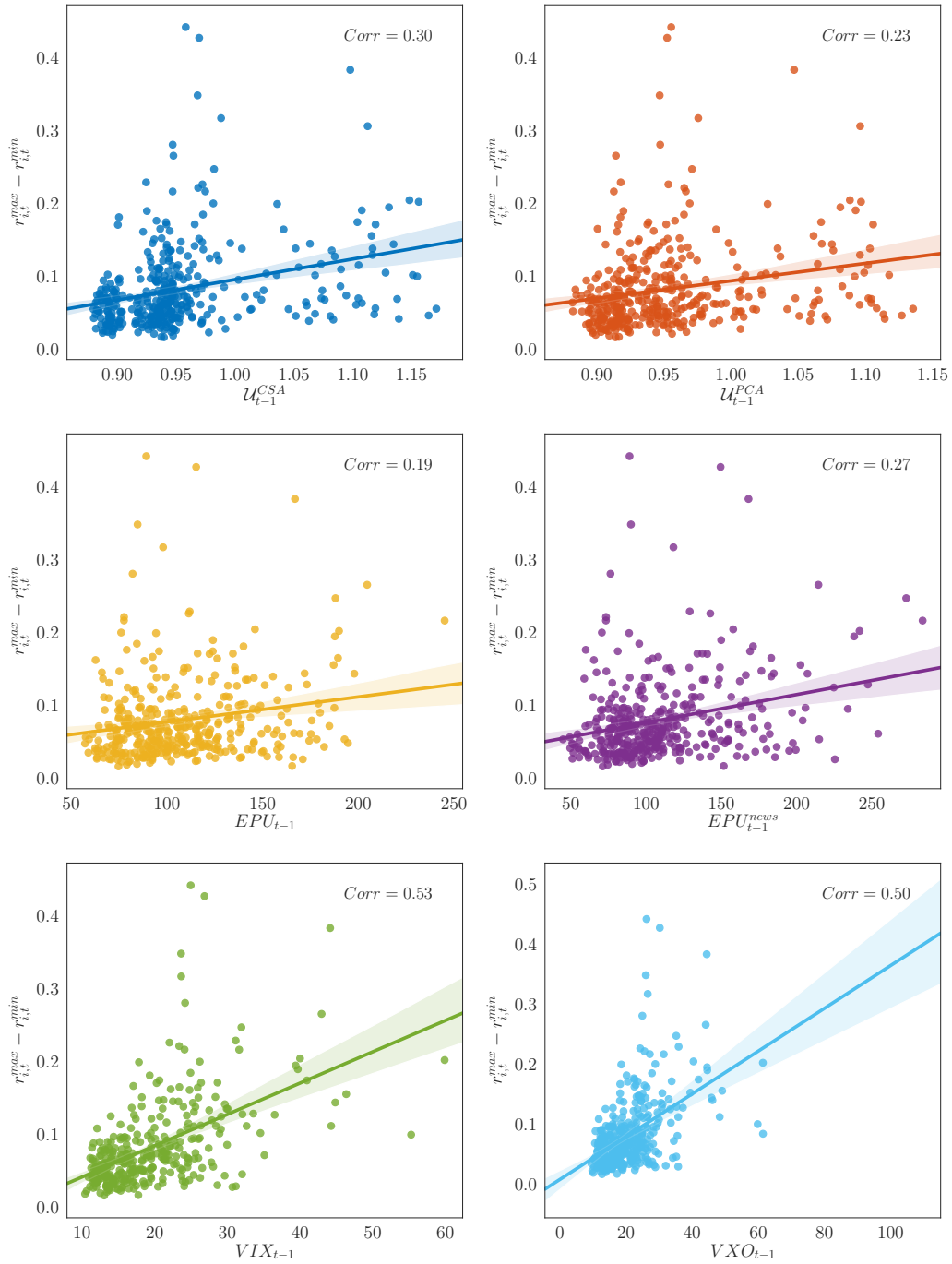


Figure A.1 Uncertainty and the Future Dispersion of Factor Returns

The dispersion is measured as the time t cross-sectional difference between maximum and minimum of factor's returns. The correlation between each uncertainty measure and dispersion is shown top-right in each panel. The uncertainty measures include the cross-sectional average (U_t^{CSA}) and first principal component (U_t^{PCA}) of uncertainties estimated using a large set of macro and financial variables from [Jurado, Ludvigson, and Ng \(2015\)](#); baseline Economic Policy Uncertainty (EPU) and news-based Economic Policy Uncertainty (EPU^{news}) from [Baker, Bloom, and Davis \(2016\)](#); CBOE stock market volatility indexes VIX and VXO .

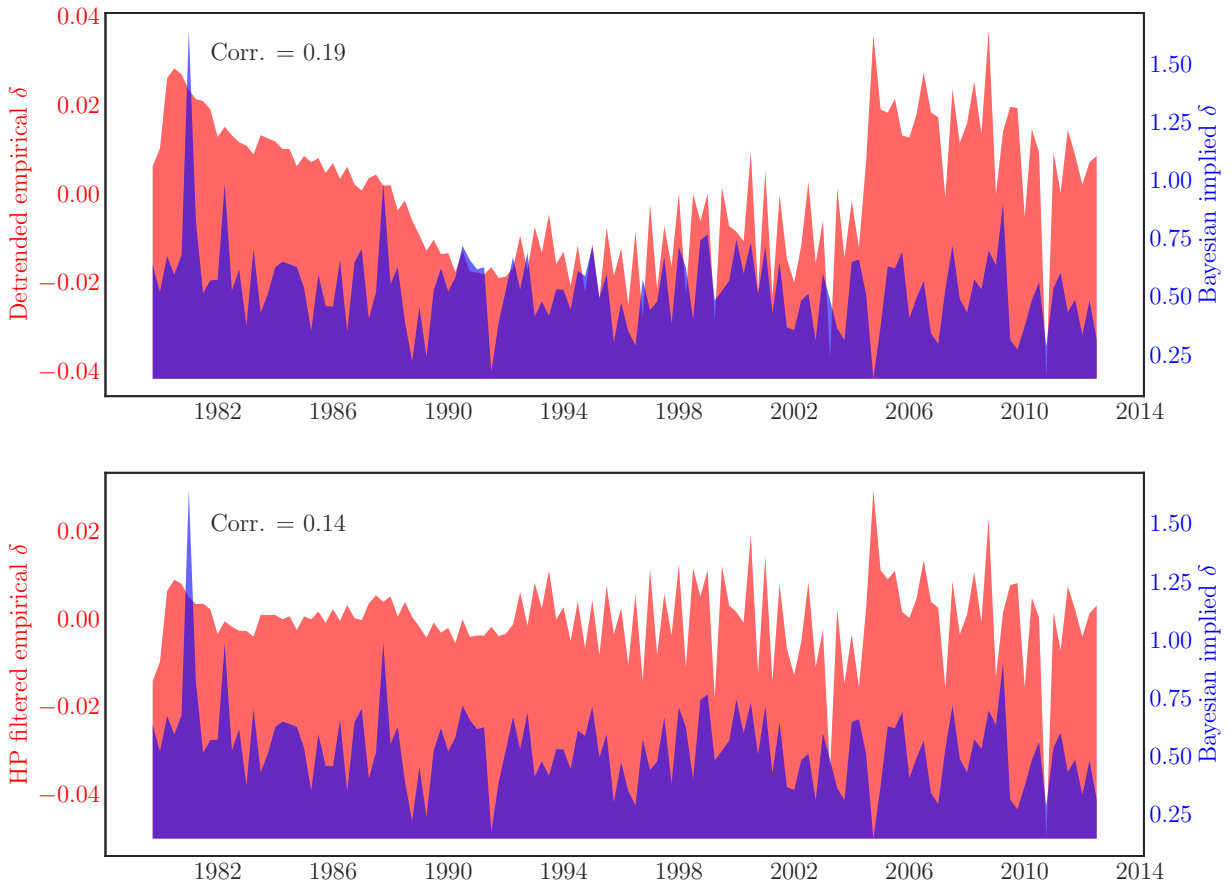


Figure A.2 The Model-implied Optimal Delegation and the Detrended Delegation in Data